On asymptotic stability of moving ground states of the nonlinear Schrödinger equation

Scipio Cuccagna

24.1.12

Abstract

We extend to the case of moving solitons, the result on asymptotic stability of ground states of the NLS obtained by the author in [Cu1]. For technical reasons we consider only smooth solutions. The proof is similar to the earlier paper. However now the flows required for the Darboux Theorem and the Birkhoff normal forms, instead of falling within the framework of standard theory of ODE's, are related to quasilinear hyperbolic symmetric systems. It is also not obvious that Darboux Theorem can be applied, since we need to compare two symplectic forms in a neighborhood of the ground states not in $H^1(\mathbb{R}^3)$, but rather in the space Σ where also the variance is bounded. But the NLS does not preserve small neighborhoods of the ground states in Σ.

1 Introduction

We consider the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}u_t = -\Delta u + \beta(|u|^2)u \quad , \quad u(0,x) = u_0(x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^3 \tag{1.1}$$

assuming: $\beta(|u|^2)u$ is "short range" and smooth; (1.1) has a smooth family of ground states.

In [Cu1] we proved that, if we break the translation invariance of the equations by either taking solutions with $u_0(-x) \equiv u_0(x)$ or by introducing some spacial inhomogeneity in the equation, for instance by adding a term $V(x)u$, the ground states are asymptotically stable, under what looks a generic hypothesis. More precisely, we assumed the sufficient conditions for orbital stability by M. Weinstein [W1]. We assumed some spectral hypotheses on the linearizations (absence of embedded eigenvalues, this probably always true under Weinstein's hypotheses) and a number of other hypotheses which hold generically (nondegeneracy of the thresholds of the continuous spectrum; some mild non resonance conditions on the eigenvalues) and which are stated in Subsect. 1.1. We then proved a form of Fermi golden rule (FGR). Specifically, we proved that certain coefficients of the system are square powers. This implies that they are non negative. We then assumed that these coefficients are in fact positive, which

is probably true generically, and used this to prove asymptotic stability of the ground states. A result similar to [Cu1], with some restrictions, is proved for the Dirac equation in [BuC].

In this paper we extend the proof of [Cu1] to equations like (1.1) without requiring any symmetry for the initial data u_0. Hence we prove that a solution $u(t)$ of (1.1) starting sufficiently close to ground states, is asymptotically as $t \nearrow +\infty$ of the form $e^{i\theta(t)+\frac{1}{2}v_+\cdot x}\phi_{\omega_+}(x-D(t))+e^{it\Delta}h_+$, for ω_+ and v_+ fixed, for $\theta \in C^1(\mathbb{R},\mathbb{R})$, $D \in C^1(\mathbb{R},\mathbb{R}^3)$ and for $h_+ \in H^1(\mathbb{R}^3)$ a small energy function. For technical reasons we need a certain known a priority regularity and decay of u_0, although in the estimates we use only the norm $\|u_0\|_{H^1}$.

The problem of stability of ground states of the NLS has a long history, partially reviewed in [Cu1]. Orbital stability was well understood in the 80's, see [CL, S, W1, GSS1, GSS2], and there is a long literature. For asymptotic stability we highlight [SW1, SW2, BP1, BP2, GS], for more references see [Cu1].

One of the crucial difficulties in asymptotic stability is in showing that the discrete modes appearing naturally in the problem, which left on their own would oscillate, are dumped by the nonlinear interaction with the continuous modes. By conservation of energy, this happens by passage of energy from the discrete to the continuous modes. The proof requires nonlinear versions of the FGR, see [Si]. In our setup, versions of the FGR of ever growing generality where proved in special cases in [BP2, SW3, TY1, TY2, TY3, Ts, BS, Cu2, GS, CM, GW]. They involved search of appropriate coordinates through Poincaré Dulac normal forms. For related linear problems, see [KW] and the references therein. [Cu3] seems to be the first reference to have noticed the relevance of the hamiltonian structure of the NLS. The intuition in [Cu3] was implemented in the series [BC, Cu1, Cu4, BuC].

The FGR consists in proving that certain coefficients are square powers, and so are generally positive. The coefficients will typically have the structure $A\cdot\overline{B}$, with A and B coefficients of the system in appropriate coordinate systems. The square power structure will follow from $B = A$. Proving such identities among the coefficients in the NLS, is certainly easier if we exploit the hamiltonian structure. We search an appropriate system of coordinates through the method of Birkhoff normal forms. In the cases considered in [BC, Cu4] this is easier because the natural coordinates which appear linearizing the system around the 0 solution, are canonical coordinates. So one can start the Birkhoff normal forms from the initial system of coordinates.

In analysis of the stability of solitons, the natural coordinates of the linearization are not canonical. Before starting the method of normal forms one has to find canonical coordinates, through an implementation of the Darboux theorem. This has to be done in the right non abstract way, in order not to lose the property that the NLS is a semilinear system. This process is done in [Cu1, BuC]. However these papers, as well as most of the papers quoted so far, avoid the case of moving solitons. Special cases, without discrete modes, were treated in [BP1, Cu5]. For multisolitons with weak interaction and no discrete modes see in [P, RSS].

Moving solitons present three special difficulties. First of all, they yield terms in the equation of the continuous modes which are non linear but which is difficult to see as perturbations of the linear equation. It is not obvious how to eliminate them through an integrating factor. Fortunately work by Beceanu, such as [Be], has solved this problem. Early solutions in particular cases are in [BP1, BP2] (see [BS, Cu5] for proofs).

The second difficulty involves the Darboux theorem. The method followed in [Cu1] becomes too complicated in the moving solitons setting. It is useful to use charge and linear momenta as coordinates. In the case of the charge, had this been done in [Cu1], it would have simplified the proof there. One difficulty with the Darboux theorem is the determination of the vectorfield $\mathcal{X}^t$ obtained as dual of an appropriate 1 form, in the Moser version of Darboux Theorem used here. In [Cu1] the existence of such $\mathcal{X}^t$ and some of its properties are rather elementary. In this paper, we are comparing two symplectic forms which are not both defined in $H^1(\mathbb{R}^3)$. Rather, they are symplectic forms in the smaller space Σ_1 formed by functions of bounded H^1 norm and bounded variance, see (1.8) and Sect.7. The proof of the existence of $\mathcal{X}^t$ would be easy if we could assume that the variance of the solutions of the NLS, assuming it is small at time $t = 0$, remained small for all times. But this is not the case, so the discussion is rather complicated.

The third difficulty present here and not in [Cu1] is that, the vector fields whose flows are used to change coordinates in the implementation of Darboux theorem and of the method of Birkhoff normal forms, do not fall as in [Cu1] within the framework of smooth vectorfields in Banach spaces. Here instead we have to deal with quasilinear hyperbolic symmetric systems. So well posedness and regularity of the flows, which in [Cu1] are elementary, are here more delicate. Particular attention requires the issue of regularity of the flows with respect to the initial data. Fortunately our systems have quite simple structure. In a rather standard way, our flows are obtained as limits of flows of systems with viscosity, which fall within the classical framework of ODE's. In the limit we lose some regularity. It is at this juncture that we use the qualitative information on regularity and decay of the initial datum u_0. The more we iterate, the more we lose regularity. Fortunately we have as much regularity and decay of u_0 and of the ground states as we want, to start with.

Acknowledgments I wish to thank G. Tondo for discussions about the reduction of variables and for pointing out reference [O].

1.1 Statement of the main result

We will assume the following hypotheses.

(H1) $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$.

(H2) There exists a $p \in (1, 5)$ such that for every $k \geq 0$ there is a fixed C_k with

$$\left| \frac{d^k}{dv^k} \beta(v^2) \right| \leq C_k |v|^{p-k-1} \quad \text{if } |v| \geq 1.$$

(H3) There exists an open interval $\mathcal{O}$ such that

$$\Delta u - \omega u + \beta(|u|^2)u = 0 \quad \text{for } x \in \mathbb{R}^3, \tag{1.2}$$

admits a C^1-family of ground states $\phi_\omega(x)$ for $\omega \in \mathcal{O}$.

(H4)

$$\frac{d}{d\omega}\|\phi_\omega\|^2_{L^2(\mathbb{R}^3)} > 0 \quad \text{for } \omega \in \mathcal{O}. \tag{1.3}$$

(H5) Let $L_+ = -\Delta + \omega - \beta(\phi_\omega^2) - 2\beta'(\phi_\omega^2)\phi_\omega^2$ be the operator whose domain is $H^2(\mathbb{R}^3)$. Then we assume that L_+ has exactly one negative eigenvalue and does not have kernel when restricted to $H^1_r(\mathbb{R}^3)$, the subspace of $H^1(\mathbb{R}^3)$ formed by functions with radial symmetry.

(H6) Let $\mathcal{H}_\omega$ be the linearized operator around ϕ_ω (see Section 2 for the precise definition). $\mathcal{H}_\omega$ has m positive eigenvalues $\lambda_1(\omega) \le \lambda_2(\omega) \le ... \le \lambda_m(\omega)$ with $0 < N_j\lambda_j(\omega) < \omega < (N_j+1)\lambda_j(\omega)$ with $N_j \ge 1$. We set $N = N_1$. Here we are repeating each eigenvalue a number of times equal to its multiplicity. We assume the multiplicity constant in ω.

(H7) There is no multi index $\mu \in \mathbb{Z}^m$ with $|\mu| := |\mu_1| + ... + |\mu_k| \le 2N_1 + 3$ such that $\mu \cdot \lambda = m$.

(H8) If $\lambda_{j_1} < ... < \lambda_{j_k}$ are k distinct λ's, and $\mu \in \mathbb{Z}^k$ satisfies $|\mu| \le 2N_1 + 3$, then we have

$$\mu_1\lambda_{j_1} + \cdots + \mu_k\lambda_{j_k} = 0 \iff \mu = 0\ .$$

(H9) $\mathcal{H}_\omega$ has no other eigenvalues except for 0 and the $\pm\lambda_j(\omega)$. The points $\pm\omega$ are not resonances.

(H10) The Fermi golden rule Hypothesis (H10) in subsection 13.2, see (13.2), holds.

(H11) We assume that u_0 is a Schwartz function.

Recall that from the ϕ_ω one can derive solitons $e^{\frac{i}{2}v\cdot x - \frac{i}{4}|v|^2t + it\omega + i\gamma}\phi_\omega(x - vt - D)$. Solutions of (1.1) starting close to a ground state, for some time can be written as

$$\begin{aligned} u(t,x) &= \tau_{D(t)}e^{i\Theta(t,x)}(\phi_{\omega(t)}(x) + r(t,x)) \\ \Theta(t,x) &= \frac{1}{2}v(t)\cdot x + \vartheta(t) \end{aligned} \tag{1.4}$$

with $\tau_D f(x) := f(x - D)$.

Theorem 1.1. *Let $\omega_1 \in \mathcal{O}$, $v_1 \in \mathbb{R}$ and $\phi_{\omega_1}(x)$ a ground state of* (1.1). *Let $u(t,x)$ be a solution to* (1.1). *Assume (H1)–(H10). Then, there exist an $\epsilon_0 > 0$ and a $C > 0$ such that if $\varepsilon := \inf_{\gamma\in\mathbb{R}, y\in\mathbb{R}^3}\|u_0 - e^{\mathrm{i}\gamma}e^{\frac{\mathrm{i}}{2}v_1\cdot x}\phi_{\omega_1}(\cdot - y)\|_{H^1} < \epsilon_0$, there exist $\omega_\pm \in \mathcal{O}$, $v_\pm \in \mathbb{R}^3$, $\theta \in C^1(\mathbb{R};\mathbb{R})$, $y \in C^1(\mathbb{R};\mathbb{R}^3)$ and $h_\pm \in H^1$ with $\|h_\pm\|_{H^1} + |\omega_\pm - \omega_1| + |v_\pm - v_1| \le C\varepsilon$ such that*

$$\lim_{t\to\pm\infty} \|u(t,\cdot) - e^{\mathrm{i}\theta(t)+\frac{\mathrm{i}}{2}v_\pm\cdot x}\tau_{y(t)}\phi_{\omega_\pm} - e^{\mathrm{i}t\Delta}h_\pm\|_{H^1} = 0. \tag{1.5}$$

In the notation of (1.4)*, we have* $\tau_{D(t)}e^{i\Theta(t,x)}r(t,x) = A(t,x) + \widetilde{r}(t,x)$ *such that* $A(t,\cdot) \in \mathcal{S}(\mathbb{R}^3,\mathbb{C})$*,* $|A(t,x)| \le C(t)$ *with* $\lim_{|t|\to\infty} C(t) = 0$ *and such that for any pair* (p,q) *which is admissible, by which we mean that*

$$2/p + 3/q = 3/2\,, \quad 6 \ge q \ge 2\,, \quad p \ge 2, \tag{1.6}$$

we have

$$\|\widetilde{r}\|_{L^p_t(\mathbb{R},W^{1,q}_x)} \le C\|u_0\|_{H^1}. \tag{1.7}$$

Remark 1.2. In the proof we use only bounds on the H^1 norm of u_0. Nonetheless we use in a qualitative fashion the fact that u_0 is quite regular and rapidly decaying. For simplicity we restrict attention to the case when u_0 is a Schwartz function.

Remark 1.3. In the proof we show that we can take $\theta = \vartheta$ and $y = D$, with (ϑ, D) the functions in Lemma 2.1. In Lemma 13.9 we show $\dot{D} = v + o(1)$ with v as in Lemma 2.1 and $\lim_{t\to\infty} o(1) = 0$. Similarly, $\dot{\vartheta} = \omega + \frac{v^2}{4} + o(1)$.

Remark 1.4. Notice that in (H6) we exclude eigenvalues of $\mathcal{H}_\omega$ in (ω,∞) mainly because we think they do not exist under (H1)–(H5). Notice that in [CPV] smoothing estimates for $\mathcal{H}_\omega$ are proved also in the presence of eigenvalues in (ω,∞), so that the theory here and in [Cu1] could be developed also in that situation.

Remark 1.5. By elementary arguments, Theorem 1.1 is a consequence of the special case where the linear momenta are equal to 0, see (2.3). So we will focus only on this case.

We briefly describe the proof, which is similar in spirit to [Cu1], but departs from [Cu1] in important ways. First of all, we need to choose a system of coordinates around the ground states. There is a natural choice related to the notion of modulation and to the spectral decomposition of the linearization. Only in a second moment we use charge and linear moment as coordinates. Since these are invariants of motion, we then consider a reduction of coordinates, in an elementary fashion. We also move to canonical coordinates through an implementation of the Darboux theorem. We then start the Birkhoff normal form argument, that is, we consider other canonical coordinates where the system looks increasingly more treatable. Finally after a finite number of them, we settle with coordinates where it is possible to prove the Fermi golden rule. Then, if (H10) is true, we conclude simultaneously that the continuous modes disperse and that the energy of the discrete modes leaks away through nonlinear interaction with the continuous modes. The most delicate and novel feature of this paper with respect to [Cu1] consists in the analysis of the flows ϕ^t used for Darboux theorem and the Birkhoff normal forms. In particular, since we are outside the realm of ODE's, it is less obvious to conclude that for fixed t the flow ϕ^t is a differentiable map. This is where (H11) is helpful. As for Birkhoff

normal forms, we also add some more material useful to understand the homological equations, which are nonlinear, and which should help to understand the analogous discussion in [Cu1], which is very succinct.

We end the introduction with some notation. Given two functions $f, g : \mathbb{R}^3 \to \mathbb{C}$ we set $\langle f|g\rangle = \int_{\mathbb{R}^3} f(x)g(x)dx$ (with no complex conjugation). Given a matrix A, we denote by by A^T, its transpose. Given two vectors A and B, we denote by $A^T B = \sum_j A_j B_j$ their inner product. Sometimes we omit the summation symbol, and we use the convention on sum over repeated indexes. For any $k, s \in \mathbb{R}$ and any Banach space K, we set

$$H^{k,s}(\mathbb{R}^3, K) = \{f : \mathbb{R}^3 \to K \text{ s.t.} \|f\|_{H^{s,k}} := \|\langle x\rangle^s \|(-\Delta + 1)^k f\|_K\|_{L^2} < \infty\}.$$

In particular we set $L^{2,s} = H^{0,s}$, $L^2 = L^{2,0}$, $H^k = H^{k,0}$. Sometimes, to emphasize that these spaces refer to spatial variables, we will denote them by $W_x^{k,p}$, L_x^p, H_x^k, $H_x^{k,s}$ and $L_x^{2,s}$. For I an interval and Y_x any of these spaces, we will consider Banach spaces $L_t^p(I, Y_x)$ with mixed norm $\|f\|_{L_t^p(I,Y_x)} := \|\|f\|_{Y_x}\|_{L_t^p(I)}$. In the course of the proof we will consider a fixed pair of spaces $H^{K,S}$ and $H^{-K,-S}$, for positive and large K and S.

We set $(\mathrm{i}\partial_x + \mathrm{i}x)^\alpha := \prod_{a=1}^3 (\mathrm{i}\partial_a + \mathrm{i}x_a)^{\alpha_a}$ for any multiindex α. For any natural number $n \geq 1$ We consider the space Σ_n defined by

$$\|U\|_{\Sigma_n}^2 := \sum_{|\alpha| \leq n} \|(\mathrm{i}\partial_x + \mathrm{i}x)^\alpha U\|_{L^2}^2 < \infty. \tag{1.8}$$

Given an operator A, we will denote by $R_A(z) = (A - z)^{-1}$ its resolvent. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will consider multi indexes $\mu \in \mathbb{N}_0^m$. For $\mu \in \mathbb{Z}^m$ with $\mu = (\mu_1, ..., \mu_m)$ we set $|\mu| = \sum_{j=1}^m |\mu_j|$. For X and Y two Banach space, we will denote by $B(X, Y)$ the Banach space of bounded linear operators from X to Y and by $B^\ell(X, Y) = B(\prod_{j=1}^\ell X, Y)$. Given a differential form α, we denote by $d\alpha$ its exterior differential.

2 Linearization and set up

Let $U = {}^t(u, \overline{u})$. We consider the energy

$$\begin{aligned} &E(U) = E_K(U) + E_P(U) \\ &E_K(U) := \int_{\mathbb{R}^3} \nabla u \cdot \nabla \overline{u} dx\,, \quad E_P(U) := \int_{\mathbb{R}^3} B(u\overline{u}) dx \end{aligned} \tag{2.1}$$

with $B(0) = 0$ and $\partial_{\overline{u}} B(|u|^2) = \beta(|u|^2)u$. We will consider the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

For $U \in H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C})$ we have the following charge and momenta, which yield invariants of motion of (1.1):

$$\begin{aligned} Q(U) &= \int_{\mathbb{R}^3} u\overline{u}dx = \frac{1}{2}\langle U|\sigma_1 U\rangle. \\ \Pi_a(U) &= \operatorname{Im}\int_{\mathbb{R}^3} \overline{u}(x)u_{x_a}(x)dx = -\frac{\mathrm{i}}{2}\langle U|\sigma_3\sigma_1\frac{\partial}{\partial x_a}U\rangle. \end{aligned} \tag{2.3}$$

Sometimes we will denote $\Pi(U) = (\Pi_1(U), \Pi_1(U), \Pi_3(U))$. We will focus only on solutions of the NLS (1.1) s.t. $\Pi(U) = 0$.

The charge Q and the momenta Π_a are in $C^\infty(H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{C})$ while $E \in C^1(H^1(\mathbb{R}^3, \mathbb{C}) \times H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{C})$. If for any such functional F we set $dF(X) = \langle \nabla F, X\rangle$ for any $X \in \mathbb{C}^2$, with dF the exterior differential and ∇F the gradient of F, then

$$\nabla Q(U) = \sigma_1 U\,, \quad \nabla \Pi_a(U) = -\mathrm{i}\sigma_3\sigma_1\frac{\partial}{\partial x_a}U. \tag{2.4}$$

For later use we set

$$\begin{aligned} &\Phi_\omega = \begin{pmatrix}\phi_\omega \\ \phi_\omega\end{pmatrix},\, q(\omega) = Q(\Phi_\omega),\, e(\omega) = E(\Phi_\omega),\, p_a(\omega) = \Pi_a(\Phi_\omega) \\ &d(\omega) = e(\omega) + \omega q(\omega). \end{aligned} \tag{2.5}$$

Equation (1.1) can be written as

$$\mathrm{i}\dot{U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix}\partial_u E \\ \partial_{\overline{u}} E\end{pmatrix} = \sigma_3\sigma_1\nabla E(U). \tag{2.6}$$

We introduce now the *linearization*

$$\begin{aligned} &\mathcal{H}_\omega := \sigma_3\sigma_1\left(\nabla^2 E(\Phi_\omega) + \omega\nabla^2 Q(\Phi_\omega)\right) = \sigma_3(-\Delta + \omega) + V_\omega \\ &\text{where } V_\omega := \sigma_3\left[\beta(\phi_\omega^2) + \beta'(\phi_\omega^2)\phi_\omega^2\right] + \mathrm{i}\sigma_2\beta'(\phi_\omega^2)\phi_\omega^2. \end{aligned} \tag{2.7}$$

The essential spectrum of $\mathcal{H}_\omega$ is $(-\infty, -\omega] \cup [\omega, +\infty)$. It is well known, [W2], that by (H4)–(H5) 0 is an isolated eigenvalue of $\mathcal{H}_\omega$ with $\dim N_g(\mathcal{H}_\omega) = 8$ and

$$\begin{aligned} &\mathcal{H}_\omega\sigma_3\Phi_\omega = 0 = \mathcal{H}_\omega\partial_{x_j}\Phi_\omega, \\ &\mathcal{H}_\omega\partial_\omega\Phi_\omega = -\Phi_\omega\,, \quad \mathcal{H}_\omega x_a\sigma_3\Phi_\omega = -\partial_{x_a}\Phi_\omega. \end{aligned} \tag{2.8}$$

Since $\mathcal{H}_\omega^* = \sigma_3\mathcal{H}_\omega\sigma_3$, we have $N_g(\mathcal{H}_\omega^*) = \operatorname{span}\{\Phi, \sigma_3\partial_\omega\Phi, \sigma_3\partial_{x_a}\Phi, x_a\Phi\}$. We consider eigenfunctions $\xi_j(\omega)$ with eigenvalue $\lambda_j(\omega)$:

$$\mathcal{H}_\omega\xi_j(\omega) = \lambda_j(\omega)\xi_j(\omega), \quad \mathcal{H}_\omega\sigma_1\xi_j(\omega) = -\lambda_j(\omega)\sigma_1\xi_j(\omega).$$

They can be normalized so that $\langle\sigma_3\mathcal{H}_\omega\xi_j(\omega), \overline{\xi}_\ell(\omega)\rangle = \delta_{j\ell}.$ Furthermore, they can be chosen to be real, that is with real entries, so $\xi_j = \overline{\xi}_j$ for all j, see Prop. 5.1 [GW].

Both ϕ_ω and $\xi_j(\omega,x)$ are smooth in $\omega\in\mathcal{O}$ and $x\in\mathbb{R}^3$ and satisfy

$$\sup_{\omega\in\mathcal{K},x\in\mathbb{R}^3} e^{a|x|}(|\partial_x^\alpha\phi_\omega(x)|+\sum_{j=1}^m|\partial_x^\alpha\xi_j(\omega,x)|)<\infty$$

for every $a\in(0,\inf_{\omega\in\mathcal{K}}\sqrt{\omega-\lambda(\omega)})$ and every compact subset $\mathcal{K}$ of $\mathcal{O}$.

For $\omega\in\mathcal{O}$, we have the $\mathcal{H}_\omega$-invariant Jordan block decomposition

$$L^2(\mathbb{R}^3,\mathbb{C}^2)=N_g(\mathcal{H}_\omega)\oplus\big(\oplus_\pm\oplus_{j=1}^m\ker(\mathcal{H}_\omega\mp\lambda_j(\omega))\big)\oplus L_c^2(\mathcal{H}_\omega), \tag{2.9}$$

$L_c^2(\mathcal{H}_\omega):=\big\{N_g(\mathcal{H}_\omega^*)\oplus\big(\oplus_{\lambda\in\sigma_d\backslash\{0\}}\ker(\mathcal{H}_\omega^*-\lambda(\omega))\big)\big\}^\perp$ with $\sigma_d=\sigma_d(\mathcal{H}_\omega)$. We also set $L_d^2(\mathcal{H}_\omega):=N_g(\mathcal{H}_\omega)\oplus\big(\oplus_{\lambda\in\sigma_d\backslash\{0\}}\ker(\mathcal{H}_\omega-\lambda(\omega))\big)$. By $P_c(\mathcal{H}_\omega)$ (resp. $P_d(\mathcal{H}_\omega)$), or simply by $P_c(\omega)$ (resp. $P_d(\omega)$), we denote the projection on $L_c^2(\mathcal{H}_\omega)$ (resp. $L_d^2(\mathcal{H}_\omega)$) associated to the above direct sum. The spectral decomposition of a vector X with respect to (2.9) is

$$\begin{aligned}&X=\big[P_{N_g(\mathcal{H}_\omega)}+\sum_{j=1}^m(P_{\ker(\mathcal{H}_\omega-\lambda_j)}+P_{\ker(\mathcal{H}_\omega+\lambda_j)})+P_c(\mathcal{H}_\omega)\big]X=\\&\frac{\langle X|\sigma_3\partial_\omega\Phi\rangle}{q'(\omega)}\sigma_3\Phi+\frac{\langle X|\Phi\rangle}{q'(\omega)}\partial_\omega\Phi+\sum_{a=1}^3\frac{\langle X|x_a\Phi\rangle}{q(\omega)}\partial_{x_a}\Phi-\sum_{a=1}^3\frac{\langle X|\sigma_3\partial_{x_a}\Phi\rangle}{q(\omega)}\sigma_3x_a\Phi\\&+\sum_{j=1}^m\langle X|\sigma_3\xi_j\rangle\xi_j+\sum_{j=1}^m\langle X|\sigma_1\sigma_3\xi_j\rangle\sigma_1\xi_j+P_c(\mathcal{H}_\omega)X.\end{aligned} \tag{2.10}$$

The following lemma is well known.

Lemma 2.1. *Fix $U_o=e^{\mathrm{i}\sigma_3(\frac{v_o\cdot(x-D_o)}{2}+\vartheta_o)}\Phi_{\omega_o}(x-D_o)$. Then $\exists$ a neighborhood $\mathcal{U}^{1,0}$ of U_o in H^1 and functions $\omega\in C^\infty(\mathcal{U}^{1,0},\mathcal{O})$, $\vartheta\in C^\infty(\mathcal{U}^{1,0},\mathbb{R})$ and $D,v\in C^\infty(\mathcal{U}^{1,0},\mathbb{R}^3)$, s.t. in U_o their value is $(\omega_o,\vartheta_o,D_o,v_o)$ and s.t. $\forall U\in\mathcal{U}^{1,0}$*

$$U(x)=e^{\mathrm{i}\sigma_3(\frac{v\cdot(x-D)}{2}+\vartheta)}(\Phi_\omega(x-D)+R(x-D))\ \text{and}\ R\in N_g^\perp(\mathcal{H}_\omega^*). \tag{2.11}$$

Notice that, once the functions are give, we have

$$R(x)=e^{-\mathrm{i}\sigma_3(\frac{v\cdot x}{2}+\vartheta)}U(x+D)-\Phi_\omega(x) \tag{2.12}$$

with the rhs just continuous in U. We can further decompose R using (2.9) as

$$R(x)=\sum_{j=1}^m z_j\xi_j(\omega,x)+\sum_{j=1}^m\overline{z}_j\sigma_1\xi_j(\omega,x)+P_c(\mathcal{H}_\omega)f(x),\quad f\in L_c^2(\mathcal{H}_{\omega_0}) \tag{2.13}$$

where we fixed $\omega_0\in\mathcal{O}$ such that $q(\omega_0)=\|u_0\|_2^2$. So we have

$$\begin{aligned}U(x)=&e^{\mathrm{i}\sigma_3(\frac{v\cdot(x-D)}{2}+\vartheta)}\big(\Phi_\omega(x-D)+z\cdot\xi(\omega,x-D)\\&+\overline{z}\cdot\sigma_1\xi(\omega,x-D)+(P_c(\mathcal{H}_\omega)f)(x-D)\big).\end{aligned} \tag{2.14}$$

(2.14) is a system of coordinates because for $\mathcal{O}$ sufficiently small the map $P_c(\mathcal{H}_\omega)$ is an isomorphism from $L_c^2(\mathcal{H}_{\omega_0})$ to $L_c^2(\mathcal{H}_\omega)$. Notice that the maps $U \to z_j$ are smooth.

We set $z \cdot \xi = \sum_j z_j \xi_j$ and $\overline{z} \cdot \sigma_1 \xi = \sum_j \overline{z}_j \sigma_1 \xi_j$. In the sequel we set

$$\partial_\omega R = \sum_{j=1}^m z_j \partial_\omega \xi_j(\omega) + \sum_{j=1}^m \overline{z}_j \sigma_1 \partial_\omega \xi_j(\omega) + \partial_\omega P_c(\mathcal{H}_\omega) f. \tag{2.15}$$

Sometimes we will denote $P_c(\omega) = P_c(\mathcal{H}_\omega)$.

We have the following formulas:

$$\begin{aligned}
&\frac{\partial}{\partial \omega} = e^{\mathrm{i}\sigma_3 \Theta} \partial_\omega (\Phi(x-D) + R(x-D)) \, , \quad \frac{\partial}{\partial \vartheta} = \mathrm{i}\sigma_3 U(x), \\
&\frac{\partial}{\partial D_a} = -\frac{\partial}{\partial x_a} U(x) \, , \quad \frac{\partial}{\partial v_a} = \frac{\mathrm{i}}{2} \sigma_3 (x_a - D_a) U(x) \text{ for } a = 1,2,3,, \\
&\frac{\partial}{\partial z_j} = e^{\mathrm{i}\sigma_3 \Theta} \xi_j (x-D) \, , \quad \frac{\partial}{\partial \overline{z}_j} = e^{\mathrm{i}\sigma_3 \Theta} \sigma_1 \xi_j (x-D) \text{ for } j = 1, ..., m.
\end{aligned} \tag{2.16}$$

Lemmas 2.2–2.4 are similar to analogous ones in [Cu1].

Lemma 2.2. *There is a matrix $\mathbb{A}$ such that*

$$\begin{pmatrix} \langle \tau_D e^{-\mathrm{i}\sigma_3 \Theta} \Phi | \\ \langle \tau_D e^{-\mathrm{i}\sigma_3 \Theta} \sigma_3 \partial_\omega \Phi | \\ 2 \langle \tau_D e^{-\mathrm{i}\sigma_3 \Theta} \sigma_3 \nabla_x \Phi | \\ \langle \tau_D e^{-\mathrm{i}\sigma_3 \Theta} x \Phi | \end{pmatrix} = \mathbb{A} \begin{pmatrix} -q'(\omega) d\omega \\ -\mathrm{i} q'(\omega) d\vartheta \\ \mathrm{i} q(\omega) dv \\ -q(\omega) dD \end{pmatrix}. \tag{2.17}$$

We have $\mathbb{A} = 1 + \mathbb{A}_1(z, f, \omega, v)$ with $\mathbb{A}_1 = (1 + |v|) O(|z| + \|f\|_{H^{-K,-S}})$ smooth in the arguments $z \in \mathbb{C}^m$, $f \in H^{-K,-S}$ and (ω, v), for any pair (K, S),.

Proof. Consider the functions of variable $(U, \omega, \vartheta, v, D)$

$$\begin{aligned}
&\mathcal{F} := \langle e^{-\mathrm{i}\sigma_3 (\frac{v \cdot (x-D)}{2} + \vartheta)} U(x) - \Phi_\omega (x-D) | \Phi_\omega (x-D) \rangle \\
&\mathcal{G} := \langle e^{-\mathrm{i}\sigma_3 (\frac{v \cdot (x-D)}{2} + \vartheta)} U(x) - \Phi_\omega (x-D) | \sigma_3 \partial_\omega \Phi_\omega (x-D) \rangle \\
&\mathcal{B}_a := \langle e^{-\mathrm{i}\sigma_3 (\frac{v \cdot (x-D)}{2} + \vartheta)} U(x) - \Phi_\omega (x-D) | (x_a - D) \Phi_\omega (x-D) \rangle \\
&\mathcal{D}_a := \langle e^{-\mathrm{i}\sigma_3 (\frac{v \cdot (x-D)}{2} + \vartheta)} U(x) - \Phi_\omega (x-D) | \sigma_3 \partial_a \Phi_\omega (x-D) \rangle .
\end{aligned} \tag{2.18}$$

Notice that differentiating in U we obtain the vectors in the lhs of (2.17), which therefore span a vector bundle which has as sections the gradients $\nabla \omega$, $\nabla \vartheta$, ∇v_a, ∇D_a. For $R(x)$ defined by (2.12), we have following partial derivatives:

$$\begin{aligned}
&\mathcal{G}_\omega = \langle R | \sigma_3 \partial_\omega^2 \Phi \rangle \, , \quad \mathcal{G}_\vartheta = \mathrm{i}(q'(\omega) + \langle R | \partial_\omega \Phi \rangle) \, , \\
&\mathcal{G}_{D_a} = \frac{\mathrm{i} v_a q'(\omega)}{2} + \langle (\partial_a + \frac{\mathrm{i}}{2} \sigma_3 v_a) R | \sigma_3 \partial_\omega \Phi \rangle \, , \quad \mathcal{G}_{v_a} = -\frac{\mathrm{i}}{2} \langle x_a R | \partial_\omega \Phi \rangle \, .
\end{aligned} \tag{2.19}$$

Similar formulas are satisfied by the other functionals in (2.18). Substituting decomposition (2.13), we see that the functions in (2.19) satisfy the regularity required for $\mathbb{A}_1$. For the other functions in (2.18) it is straightforward to check that the same is true. This yields by an elementary argument Lemma 2.2. $\square$

Lemma 2.3. *For $\mathcal{U}^{1,0}$ in Lemma 2.1 sufficiently small, $z_j \in C^\infty(\mathcal{U}^{1,0}, \mathbb{C})$. The following formulas hold, summing over repeated index a:*

$$\begin{aligned}\nabla z_j =& - \langle \sigma_3 \xi_j | \partial_\omega R \rangle \nabla \omega - \mathrm{i} \langle \sigma_3 \xi_j | \sigma_3 R \rangle \nabla \vartheta - \frac{\mathrm{i}}{2} \langle \sigma_3 \xi_j | \sigma_3 x_a R \rangle \nabla v_a \\ &+ \langle \sigma_3 \xi_j | (\partial_{x_a} + \mathrm{i} \sigma_3 \frac{v_a}{2}) R \rangle \nabla D_a + \tau_D (e^{-\mathrm{i}\sigma_3 \Theta} \sigma_3 \xi_j)(x) \\ \nabla \overline{z}_j =& - \langle \sigma_1 \sigma_3 \xi_j | \partial_\omega R \rangle \nabla \omega - \mathrm{i} \langle \sigma_1 \sigma_3 \xi_j | \sigma_3 R \rangle \nabla \vartheta - \frac{\mathrm{i}}{2} \langle \sigma_1 \sigma_3 \xi_j | \sigma_3 x_a R \rangle \nabla v_a \\ &+ \langle \sigma_1 \sigma_3 \xi_j | (\partial_{x_a} + \mathrm{i} \sigma_3 \frac{v_a}{2}) R \rangle \nabla D_a + \tau_D (e^{-\mathrm{i}\sigma_3 \Theta} \sigma_1 \sigma_3 \xi_j)(x).\end{aligned}$$

Proof. The fact that $z_j \in C^\infty(\mathcal{U}^{1,0}, \mathbb{C})$ follows from formula

$$z_j(U) = \langle U(x), e^{-\mathrm{i}\sigma_3 (\frac{v\cdot(x-D)}{2} + \vartheta)} \sigma_3 \xi_j(\omega, x - D) \rangle,$$

the fact that $\omega, \vartheta, v, D \in C^\infty(\mathcal{U}^{1,0})$ and the properties of $\xi_j(\omega, x)$. We have

$$\begin{aligned}&\langle \nabla z_j | \tau_D e^{\mathrm{i}\sigma_3 \Theta} \xi_\ell \rangle = \delta_{j\ell}, \quad \langle \nabla z_j | \tau_D e^{\mathrm{i}\sigma_3 \Theta} \sigma_1 \xi_\ell \rangle = 0 = \langle \nabla z_j | \tau_D e^{\mathrm{i}\sigma_3 \Theta} \sigma_3 (\Phi + R) \rangle \\ &\langle \nabla z_j | \tau_D e^{\mathrm{i}\sigma_3 \Theta} \partial_\omega (\Phi + R) \rangle = 0 = \langle \nabla z_j, \tau_D e^{\sigma_3 \Theta} x_a (\Phi + R) \rangle \\ &\langle \nabla z_j | \tau_D e^{\sigma_3 \Theta} \left(\partial_{x_a} + \mathrm{i} \sigma_3 \frac{v_a}{2} \right) (\Phi + R) \rangle = 0 \\ &\langle \nabla z_j | \tau_D e^{\mathrm{i}\sigma_3 \Theta} P_c(\omega) P_c(\omega_0) g \rangle = 0 \quad \forall g \in L^2_c(\mathcal{H}_{\omega_0}).\end{aligned} \tag{2.20}$$

The latter implies $P_c(\mathcal{H}^*_{\omega_0}) P_c(\mathcal{H}^*_\omega) e^{\mathrm{i}\sigma_3 \Theta} \tau_{-D} \nabla z_j = 0$ and $P_c(\mathcal{H}^*_\omega) e^{\mathrm{i}\sigma_3 \Theta} \tau_{-D} \nabla z_j = 0$. By (2.10) and for $\mathbf{A}$ and unknown vector,

$$\nabla z_j = \mathbf{A}^T \begin{pmatrix} \tau_D((e^{-\mathrm{i}\sigma_3 \Theta} \Phi)(x) \\ \tau_D(e^{-\mathrm{i}\sigma_3 \Theta} \sigma_3 \partial_\omega \Phi)(x) \\ \tau_D(e^{-\mathrm{i}\sigma_3 \Theta} x \Phi)(x) \\ \tau_D(e^{-\mathrm{i}\sigma_3 \Theta} \sigma_3 \nabla_x \Phi)(x) \end{pmatrix} + \tau_D(e^{-\mathrm{i}\sigma_3 \Theta} \sigma_3 \xi_j)(x).$$

Using Lemma 2.2 there is a vector $\mathbf{A}_1$ s.t.

$$\nabla z_j = \mathbf{A}_1^T \begin{pmatrix} \nabla \omega \\ \nabla \vartheta \\ \nabla v \\ \nabla D \end{pmatrix} + \tau_D(e^{-\mathrm{i}\sigma_3 \Theta} \sigma_3 \xi_j)(x).$$

By (2.16) and (2.20) we obtain

$$\mathbf{A}_1 = - \begin{pmatrix} \langle \sigma_3 \xi_j | \partial_\omega R \rangle \\ \mathrm{i} \langle \sigma_3 \xi_j | \sigma_3 R \rangle \\ \frac{\mathrm{i}}{2} \langle \sigma_3 \xi_j | \sigma_3 x R \rangle \\ - \langle \sigma_3 \xi_j | (\nabla_x + \mathrm{i} \sigma_3 \frac{v}{2}) R \rangle \end{pmatrix}.$$

Similar formulas hold for $\nabla \overline{z}_j$ yielding Lemma 2.3. $\square$

Lemma 2.4. *The map $U \to f(U) = f$, for U and f as in* (2.14)*, is continuous from X into itself, for $X = L^2, H^1$ and Σ_n for any n. Furthermore we have $f \in C^1(\mathcal{U}^{1,0}, L^{2,-1})$, $f \in C^1(\mathcal{U}^{1,0} \cap \Sigma, L^2)$ and $f \in C^1(\mathcal{U}^{1,0} \cap \Sigma_n, \Sigma_{n-1})$ with Frechét derivative $f'(U)$ defined by the following formula, summing on the repeated index a,*

$$\begin{aligned} f'(U) = (P_c(\omega)P_c(\omega_0))^{-1} P_c(\omega)\big[-\partial_\omega R\, d\omega \\ - \mathrm{i}\sigma_3 R\, d\vartheta - \frac{\mathrm{i}}{2}\sigma_3 x_a R\, dv_a + (\partial_{x_a} + \mathrm{i}\sigma_3 \frac{v_a}{2}) R\, dD_a + e^{-\mathrm{i}\sigma_3\Theta}\tau_{-D}\ 1\!\!1\big]. \end{aligned} \tag{2.21}$$

We further have $f \in C^n(\mathcal{U}^{1,0} \cap \Sigma, H^{-n+1,-n})$

Proof. Continuity follows from Lemmas 2.1-2.3 and formula (2.14) solved w.r.t. f. The latter proves also the C^1 as well the C^n properties. The proof of formula (2.21) is similar to that of Lemma 2.3, see also Lemma 4.2 [Cu1]. $\square$

In the sequel given a scalar function $\psi(U)$ which is differentiable, we will denote by $\nabla_f \psi(U)$ the only element in $L^2_c(\mathcal{H}^*_{\omega_0})$ s.t. for any $g \in L^2_c(\mathcal{H}_{\omega_0})$ we have $\langle \nabla_f \psi(U)|g\rangle = \langle \nabla\psi(U)|P^*_c(\omega)g\rangle$.

3 Symplectic structure

Our ambient space is $\mathbf{X} \times \mathbf{X}$ where we can have $\mathbf{X} = L^2, H^1, \Sigma_n$. We focus only on points with $\sigma_1 U = \overline{U}$, so that the space is identified with $\mathbf{X}$. The natural symplectic structure for our problem is

$$\Omega(X, Y) = \langle X|\sigma_3\sigma_1 Y\rangle. \tag{3.1}$$

Definition 3.1. Let $F \in C^1(\mathcal{U}, \mathbb{C})$ for $\mathcal{U}$ and open subset of $\mathbf{X}$. Then the Hamiltonian vectorfield of F with respect to a symplectic form Ω is the field X_F such that $\Omega(X_F, Y) = -\mathrm{i}dF(Y)$ for any given tangent vector $Y \in T\mathcal{U}$. More explicitly, $X_F = -\mathrm{i}\sigma_3\sigma_1\nabla F$ for the form in (3.1).

Definition 3.2. For $F, G \in C^1(\mathcal{U}, \mathbb{C})$ as above, we call Poisson bracket of a pair of F and G the function

$$\{F, G\} = dF(X_G) = \mathrm{i}\Omega(X_F, X_G) = -\mathrm{i}\langle \nabla F|\sigma_3\sigma_1\nabla G\rangle. \tag{3.2}$$

Let $\mathcal{G} \in C^1(\mathcal{U}, \mathbb{E})$ with $\mathbb{E}$ a given Banach space on $\mathbb{C}$. Then, for $F \in C^1(\mathcal{U}, \mathbb{C})$ we set, for $\mathcal{G}'$ the Frechet derivative of $\mathcal{G}$,

$$\{\mathcal{G}, F\} := \mathcal{G}'(U)X_F(U) = -\mathrm{i}\mathcal{G}'(U)\sigma_3\sigma_1\nabla F(U). \tag{3.3}$$

We set $\{F, \mathcal{G}\} := -\{\mathcal{G}, F\}$.

Obviously our system is hamiltonian. It is important to cast it in terms of the Poisson brackets.

Lemma 3.3. *In the coordinate system* (2.14)*, system* (2.6) *can be written, for* $F = E$ *as* $\dot{\vartheta} = \{\vartheta, F\}$ *and*

$$\begin{aligned} &\dot{\omega} = \{\omega, F\}\,, \quad \dot{f} = \{f, F\}\,, \\ &\dot{D}_a = \{D_a, F\}\,, \quad \dot{v}_a = \{v_a, F\} \text{ for } a = 1, 2, 3\,, \\ &\dot{z}_j = \{z_j, F\}\,, \quad \dot{\overline{z}}_j = \{\overline{z}_j, F\} \text{ for } j = 1, ..., m\,. \end{aligned} \tag{3.4}$$

Proof. The statement is not standard only for $\dot{f} = \{f, E\}$. Notice that (2.6) can be written as

$$\begin{aligned} &\mathrm{i}\dot{U} = -\sigma_3 \left(\dot{\vartheta} + \frac{\dot{v} \cdot (x - D)}{2} \right) U + \mathrm{i}\dot{\omega} e^{\mathrm{i}\sigma_3 \Theta} \partial_\omega (\Phi + R) \\ &- \mathrm{i}\dot{D} \cdot \nabla(^{\mathrm{i}\sigma_3 \Theta}(\Phi + R)) + \mathrm{i}e^{\mathrm{i}\sigma_3 \Theta}(\dot{z} \cdot \xi + \dot{\overline{z}} \cdot \sigma_1 \xi + P_c(\mathcal{H}_\omega)\dot{f}). \end{aligned} \tag{3.5}$$

So, by (2.16), system $\mathrm{i}\dot{U} = \sigma_3 \sigma_1 \nabla E(U)$ is the same as

$$\begin{aligned} &\mathrm{i}\dot{\vartheta} \frac{\partial}{\partial \vartheta} + \sum_{a=1}^{3} \dot{v}_a \frac{\partial}{\partial v_a} + \mathrm{i}\dot{\omega} \frac{\partial}{\partial \omega} + \sum_{a=1}^{3} \dot{D}_a \frac{\partial}{\partial D_a} + \mathrm{i} \sum_{j=1}^{m} \dot{z}_j \frac{\partial}{\partial z_j} + \mathrm{i} \sum_{j=1}^{m} \dot{\overline{z}}_j \frac{\partial}{\partial \overline{z}_j} \\ &+ \mathrm{i}e^{\mathrm{i}\Theta\sigma_3} P_c(\mathcal{H}_\omega)\dot{f} = \sigma_3 \sigma_1 \nabla E(U). \end{aligned} \tag{3.6}$$

When we apply the derivative $f'(U)$ to (3.6) the first line cancels, so we get

$$f'(U) e^{\mathrm{i}\Theta\sigma_3} P_c(\mathcal{H}_\omega)\dot{f} = -f'(U)\mathrm{i}\sigma_3\sigma_1 \nabla E(U) = f'(U) X_E(U) = \{f, E\},$$

from the definition of hamiltonian field and of Poisson bracket. Notice now that $\dot{f} = f'(U)e^{\mathrm{i}\Theta\sigma_3} P_c(\mathcal{H}_\omega)\dot{f}$. This follows from the fact that

$$\begin{aligned} &f'(U)e^{\mathrm{i}\Theta\sigma_3} P_c(\mathcal{H}_\omega)\dot{f} = \frac{d}{ds}_{|s=0} f(U(\omega, \vartheta, D, v, z, \overline{z}, f + s\dot{f})) \\ &= \frac{d}{ds}_{|s=0} (f + s\dot{f}) = \dot{f}. \end{aligned}$$

Hence $\dot{f} = \{f, E\}$. □

4 Reduction of variables

The following formulas are important.

Lemma 4.1. *Consider charge* Q *and momenta* Π_a*, see defined in* (2.3)*. Then*

$$X_Q = -\frac{\partial}{\partial \vartheta}\,, \quad X_{\Pi_a} = -\frac{\partial}{\partial D_a}. \tag{4.1}$$

In particular

$$\{Q, \vartheta\} = \{\Pi_a, D_a\} = 1 \tag{4.2}$$

and any Poisson bracket not in (4.2) *of any of the invariants* Q, Π_a *with any of the coordinates from* (2.14) *is equal to 0. The functions* Q*,* Π_a *and* E *Poisson commute.*

Proof. (4.1) follows from (2.16):

$$X_Q(U) = -\mathrm{i}\sigma_3\sigma_1\nabla Q(U) = -\mathrm{i}\sigma_3 U = -\frac{\partial}{\partial\vartheta}\,,$$
$$X_{\Pi_a}(U) = -\mathrm{i}\sigma_3\sigma_1\nabla\Pi_a(U) = \mathrm{i}\sigma_3\sigma_1\sigma_2\partial_a U = \partial_a U = -\frac{\partial}{\partial D_a}.$$

The second part of the statement follows immediately from (4.1). $\square$

The following lemma is elementary.

Lemma 4.2. *For $U = e^{\mathrm{i}\sigma_3(\frac{v\cdot(x-D)}{2}+\vartheta)}(\Phi_\omega(x-D)+R(x-D))$ and summing on repeated indexes, we have the following formulas:*

$$Q(U) = Q(\Phi_\omega + R) = q(\omega) + Q(R)\,;$$
$$\Pi_a(U) = \Pi_a(\Phi_\omega + R) - 2^{-1}v_a Q(\Phi_\omega + R)\,;$$
$$E(U) = E(\Phi_\omega + R) - v_a\Pi_a(\Phi_\omega + R) + \frac{v^2}{4}Q(\Phi_\omega + R).$$

We have $\Pi_a(\Phi_\omega + R) = \Pi_a(R)$.
$Q(U)$ and $\Pi_a(U)$ are C^∞ and $E(U)$ is C^1 in ω, v, z and $f \in H^1$. Furthermore both E and ∇E depend smoothly on (ω, z).

We introduce now a new hamiltonian, for $U_0^T = (u_0, \overline{u}_0)$,

$$K(U) := E(U) - E\left(\Phi_{\omega_0}\right) + \omega(U)\left(Q(U) - Q(U_0)\right). \tag{4.3}$$

Lemma 4.1 implies that the solutions of (2.6) with charge $q(\omega_0)$ satisfy (3.4) with $F = K$ and $\dot\vartheta - \omega = \{\vartheta, K\}$. We would like for K to be our hamiltonian, but obviously K is not a hamiltonian for (2.6). To obviate this we notice that $\partial_{D_a}K = \partial_\vartheta K \equiv 0$ imply that the evolution of the variables $\omega, v_a, z_j, \overline{z}_j, f$ is unchanged if we consider the following new hamiltonian system:

$$\begin{aligned}
&\dot\omega = \{\omega, K\}\,, \quad \dot\vartheta = \{\vartheta, K\}\,, \quad \dot f = \{f, K\}\,,\\
&\dot D_a = \{D_a, K\}\,, \quad \dot v_a = \{v_a, K\} \text{ for } a = 1,2,3\,,\\
&\dot z_j = \{z_j, K\}\,, \quad \dot{\overline{z}}_j = \{\overline{z}_j, K\} \text{ for } j = 1, ..., m\,.
\end{aligned} \tag{4.4}$$

It is elementary that for (4.4) the charge $Q(U)$ and the momenta $\Pi_a(U)$ are invariants of motion.

We proceed to a reduction of order in (4.4) such as as described for instance in Theorem 6.35 p.402 [O]. Here the discussion is elementary because we have no need to prove the existence of particular variables, see p.395 [O]. We set

$$\begin{aligned}
&Q := Q(U) = q(\omega) + Q(R) \quad ,\\
&\Pi_a := \Pi_a(U) = \Pi_a(R) - \frac{v_a}{2}\left(q(\omega) + Q(R)\right).
\end{aligned} \tag{4.5}$$

Lemma 4.3. *After we express R in (4.5) in terms of ω, z and f, see (2.13), there is an implicit function $\omega = \omega(Q, z, f)$ defined by the first of (4.5), with $\omega(Q, z, f)$ smooth in Q, z and in $f \in L^2_c(\omega_0)$. Similarly, $v_a = v_a(\Pi_a, Q, z, f)$, with the latter smooth in Π_a, Q, z and $f \in H^{\frac{1}{2}}_c(\omega_0)$.*

Proof. For $\omega = \omega(Q, z, f)$ the statement follows from the implicit function theorem . Write $v_a = 2Q^{-1}(\Pi_a(R) - \Pi_a)$ and substitute $\omega = \omega(Q, z, f)$ in the definition of R. $\square$

Lemma 4.3 allows to move from the variables in the rhs of (2.14) to a new system of variables obtained replacing the functions (ω, v) with the (Q, Π).

Lemma 4.4. *The vectorfields $\frac{\partial}{\partial \vartheta}$ and $\frac{\partial}{\partial D_a}$ are the same for the two systems of coordinates. In particular, in the new system of coordinates we continue to have $\frac{\partial}{\partial \vartheta} K = \frac{\partial}{\partial D_a} K = 0$.*

Proof. It is an immediate consequence of $\frac{\partial}{\partial \vartheta} Q(U) = \frac{\partial}{\partial D_a} Q(U) = 0$ and of $\frac{\partial}{\partial \vartheta} \Pi_b(U) = \frac{\partial}{\partial D_a} \Pi_b(U) = 0$ in the old coordinate system, and of the chain rule. $\square$

In the new variables, system (4.4) reduces to the pair of systems

$$\dot{Q} = 0\,, \quad \dot{\vartheta} = \{\vartheta, K\}\,, \quad \dot{D}_a = \{D_a, K\}\,, \quad \dot{\Pi}_a = 0 \quad \text{for } a = 1, 2, 3\,, \tag{4.6}$$

and

$$\dot{f} = \{f, K\}\,, \quad \dot{z}_j = \{z_j, K\}\,, \quad \dot{\overline{z}}_j = \{\overline{z}_j, K\} \quad \text{for } j = 1, ..., m\,. \tag{4.7}$$

Now we restrict to the set with $Q = Q(U_0)$ and $\Pi_a = 0$. Thanks to Lemma 4.4 system (4.7) is closed.

Lemma 4.5. *Consider the restriction of the variables (Q, Π) at the fixed values $Q = Q(U_0)$ and $\Pi_a = 0$ and set $\varrho(f) := (Q(f), \Pi(f))$. Then we have the expansion*

$$K = \underline{\psi}(\varrho(f)) + K_2 + \underline{\mathcal{R}}^{(1)} \tag{4.8}$$

where $\underline{\psi}(\varrho)$ is smooth in ϱ and where the following holds.

(1) We have

$$K_2 = \sum_{\substack{|\mu+\nu|=2 \\ \lambda^0 \cdot (\mu-\nu)=0}} \underline{a}_{\mu\nu}(\varrho(f)) z^\mu \overline{z}^\nu + \frac{1}{2} \langle \sigma_3 \mathcal{H}_{\omega_0} f | \sigma_1 f \rangle. \tag{4.9}$$

(2) We have $\underline{\mathcal{R}}^{(1)} = \widetilde{\underline{\mathcal{R}}^{(1)}} + \widetilde{\underline{\mathcal{R}}^{(2)}}$*, with* $\widetilde{\mathcal{R}^{(1)}} =$

$$= \sum_{\substack{|\mu+\nu|=2 \\ \lambda^0\cdot(\mu-\nu)\neq 0}} \underline{a}_{\mu\nu}(\varrho(f)) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=1} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 \underline{G}_{\mu\nu}(\varrho(f)) | f\rangle \tag{4.10}$$

$$\begin{aligned} \widetilde{\underline{\mathcal{R}}^{(2)}} &= \sum_{|\mu+\nu|=3} z^\mu \overline{z}^\nu \underline{a}_{\mu\nu}(z,\varrho(f)) + \sum_{|\mu+\nu|=2} z^\mu \overline{z}^\nu \langle \underline{G}_{\mu\nu}(z,\varrho(f)) | \sigma_3\sigma_1 f\rangle \\ &+ \sum_{d=2}^{4} \langle \underline{B}_d(z,\varrho(f)) | f^d\rangle + \int_{\mathbb{R}^3} \underline{B}_5(x,z,f(x),\varrho(f)) f^5(x) dx + E_P(f). \end{aligned}$$

with $\underline{B}_2(0,0) = 0$ *and where, both here and in Theorem 12.1 later, by* $f^d(x)$ *we schematically represent* $d-$*products of components of* f*.*

(3) At $\varrho(f) = 0$

$$\begin{aligned} &\underline{a}_{\mu\nu}(0) = 0 \text{ for } |\mu+\nu| = 2 \text{ with } (\mu,\nu) \neq (\delta_j,\delta_j) \text{ for all } j, \\ &\underline{a}^{(r)}_{\delta_j\delta_j}(0) = \lambda_j(\omega_0), \text{ where } \delta_j = (\delta_{1j}, ..., \delta_{mj}), \\ &\underline{G}_{\mu\nu}(0) = 0 \text{ for } |\mu+\nu| = 1 \end{aligned} \tag{4.11}$$

These $\underline{a}_{\mu\nu}(\varrho)$ *and* $\underline{G}_{\mu\nu}(x,\varrho)$ *are smooth in all variables with* $G_{\mu\nu}(\cdot,\varrho) \in C^\infty(\mathbb{R}^4, H^{K,S}_x(\mathbb{R}^3,\mathbb{C}^2))$ *for all* (K,S)*.*

(4) $\underline{a}_{\mu\nu}(z,\varrho) \in C^\infty(\mathrm{U},\mathbb{C})$ *for a small neighborhood* U *of* $(0,0)$ *in* $\mathbb{C}^m \times \mathbb{R}^4$*.*

(5) $\underline{G}_{\mu\nu}(\cdot,z,\varrho) \in C^\infty(\mathrm{U}, H^{K,S}_x(\mathbb{R}^3,\mathbb{C}^2))$*, for* U *like in (4) and for all* (K,S)*.*

(6) $\underline{B}_d(\cdot,z,\varrho) \in C^\infty(\mathrm{U}, H^{K,S}_x(\mathbb{R}^3, B((\mathbb{C}^2)^{\otimes d},\mathbb{C})))$*, for* $2 \le d \le 4$ *for* U *and* (K,S) *like above.*

(7) Let ${}^t\eta = (\zeta,\overline{\zeta})$ *for* $\zeta \in \mathbb{C}$*. Then for* $\underline{B}_5(\cdot,\omega,z,\eta,\varrho)$ *we have for all* (K,S)

$$\text{for any } l \text{ , } \|\nabla^l_{z,\overline{z},\zeta,\overline{\zeta},\varrho} \underline{B}_5(z,\eta,\varrho)\|_{H^{K,S}_x(\mathbb{R}^3, B((\mathbb{C}^2)^{\otimes 5},\mathbb{C})} \le C_l(K,S).$$

(8) For all indexes and for $b_{\mu\nu} = \underline{a}_{\mu\nu}$ *and* $B_{\mu\nu} = \underline{G}_{\mu\nu}$*, we have:*

$$b_{\mu\nu} = \overline{b_{\nu\mu}}\,, \quad B_{\mu\nu} = -\sigma_1 \overline{B_{\nu\mu}}. \tag{4.12}$$

(9) $\underline{B}_5$ *depends on* $f(x)$ *with terms* $|f(x)|^2$ *and* $\varphi(x)f(x)$*, with* $\varphi(x)$ *Schwartz functions.*

Proof. By Lemma 4.2 we get from (4.3), for $d(\omega)$ and $q(\omega)$ see (2.5),

$$\begin{aligned} K(U) &= E(\Phi_\omega + R) + \omega Q(\Phi_\omega + R) - d(\omega_0) - (\omega-\omega_0) q(\omega_0) \\ &+ \frac{v^2}{4} Q(\Phi_\omega + R) - v_a \Pi_a(R). \end{aligned} \tag{4.13}$$

We have

$$E(\Phi_\omega + R) + \omega Q(\Phi_\omega + R) = d(\omega) + \frac{1}{2}\langle \sigma_3 \mathcal{H}_\omega R | \sigma_1 R\rangle + \widehat{\mathcal{R}}^{(1)}, \tag{4.14}$$

with an expansion of the following form:

$$\widehat{\mathcal{R}}^{(1)} = \sum_{|\mu+\nu|=3} z^\mu \overline{z}^\nu \underline{a}_{\mu\nu}(\omega, z) + \sum_{|\mu+\nu|=2} z^\mu \overline{z}^\nu \langle \underline{G}_{\mu\nu}(\omega, z) | \sigma_3 \sigma_1 P_c(\omega) f\rangle$$
$$+ \sum_{d=2}^{4} \langle \underline{B}_d(\omega, z) | f^d\rangle + \int_{\mathbb{R}^3} \underline{B}_5(x, \omega, z, f(x)) f^5(x) dx + E_P(P_c(\omega) f).$$

The coefficients in the expansion of $\widehat{\mathcal{R}}^{(1)}$ satisfy appropriate smoothness and symmetry properties. When we restrict to $Q = Q(U_0)$, by $q(\omega) + Q(R) = q(\omega_0)$ we get that $\omega - \omega_0 = O(|z|^2 + \|f\|_2^2 + \|f\|^2_{H^{-K,-S}})$. Notice that we can express the function $\omega(Q, z, f)$ as a function $\omega(Q, z, f, \varrho_0(f))$, with $\varrho_0(f) = Q(f)$, such that $\omega(Q, z, f, \varrho_0)$ is smooth in the variables $z \in \mathbb{C}^m$, $f \in H^{-K,-S}$ and $Q, \varrho_0 \in \mathbb{R}$. Hence $\widehat{\mathcal{R}}^{(1)}$ is a sum of terms which can be absorbed in $\widetilde{\mathcal{R}^{(2)}}$, with dependence on $\varrho(f) = (Q(f), \Pi(f))$ reduced at that on $Q(f)$. We have by $d' = q$

$$d(\omega) - d(\omega_0) - (\omega - \omega_0) q(\omega_0) = \frac{1}{2} d''(\omega_0)(\omega - \omega_0)^2 + o(\omega - \omega_0)^2. \tag{4.15}$$

So (4.15) can be absorbed in part in $\underline{\psi}(\varrho(f))$ and part in the rest of expansion (4.8). We have

$$\begin{aligned} \frac{1}{2}\langle \sigma_3 \mathcal{H}_\omega R | \sigma_1 R\rangle &= \frac{1}{2}\langle \sigma_3 \mathcal{H}_{\omega_0} R | \sigma_1 R\rangle \\ &\quad + (\omega - \omega_0) Q(R) + \frac{1}{2}\langle \sigma_3 (V_\omega - V_{\omega_0}) R | \sigma_1 R\rangle, \end{aligned} \tag{4.16}$$

where $\frac{1}{2}\langle \sigma_3 \mathcal{H}_{\omega_0} R | \sigma_1 R\rangle = \sum_j \lambda_j(\omega_0)|z_j|^2 + \frac{1}{2}\langle \sigma_3 \mathcal{H}_{\omega_0} f | \sigma_1 f\rangle$. In the last line of (4.16), the first term is similar to (4.15) and the second can be absorbed in $H_2^{(1)} + \mathcal{R}^{(1)}$. Since we have $v_a q(\omega_0) = 2\Pi_a(R)$, we get that also the second line of (4.13) can be decomposed into terms with the properties of the various terms in the rhs of (4.8).

□

5 A vectorfield needed for Darboux Theorem

We introduce the 2-form, summing on repeated indexes,

$$\Omega_0 = \mathrm{i} d\vartheta \wedge dQ + \mathrm{i} dD_a \wedge d\Pi_a + dz_j \wedge d\overline{z}_j + \langle f' | \sigma_3 \sigma_1 f'\rangle. \tag{5.1}$$

Through a change of variable we will transform the Ω defined (3.1) into the Ω_0. A brief reminder of J.Moser's scheme of proof is in Sect.7 [Cu1] and will not be

repeated here. The first step in the implementation of this scheme is the search of an appropriate vectorfield $\mathcal{X}^t$, which satisfies an equation stated in Lemma 5.2. Then we will have $\phi^*\Omega = \Omega_0$, with $\phi = \phi^1$ the Lie transform associated to ϕ^t, the flow of $\mathcal{X}^t$. Existence and differentiability of ϕ^t are not obvious. Lemma 5.8 will tell us that to get ϕ^t we need information on a quasilinear hyperbolic symmetric system mixed to ODE's for the discrete modes. In Lemma 5.8 we will establish some properties of the coefficients of the system. Sections 6–8 contain material needed to establish existence and differentiability of ϕ^t. Lemma 5.1 is similar to Lemma 7.2 [Cu1].

Lemma 5.1. *At the points $\tau_D e^{\mathrm{i}\sigma_3\left(\frac{v\cdot x}{2}+\vartheta\right)}\Phi_{\omega_0}$ for all $(\vartheta, D, v) \in \mathbb{R}^7$ we have $\Omega_0 = \Omega$.*
Consider the following forms:

$$\mathrm{B}_0 := \mathrm{i}\vartheta dQ + \mathrm{i}D_a d\Pi_a - \frac{\overline{z}_j dz_j - z_j d\overline{z}_j}{2} + \frac{1}{2}\langle f|\sigma_3\sigma_1 f'\rangle \tag{5.2}$$

and $\mathrm{B} := \mathrm{B}_0 + \Gamma$ *for* $\Gamma = \varphi dQ + (\Gamma)_j dz_j + (\Gamma)_{\overline{j}} d\overline{z}_j + \langle (\Gamma)_f | f'\rangle$, *where*

$$\begin{aligned}
&\varphi := \frac{\langle \sigma_1\sigma_3 R|\partial_\omega R\rangle + \mathrm{i}v_a\langle x_a\sigma_1 R|\partial_\omega R\rangle}{2(q' + \partial_\omega Q(R))}, \\
&(\Gamma)_j := \frac{\mathrm{i}}{2}v_a\langle x_a\sigma_1 R|\xi_j\rangle - \varphi\partial_j Q(R)\,,\ (\Gamma)_{\overline{j}} := \frac{\mathrm{i}}{2}v_a\langle x_a R|\xi_j\rangle - \varphi\partial_{\overline{j}} Q(R), \\
&(\Gamma)_f := P_c^*(\omega_0)\left(\frac{\mathrm{i}}{2}v_a P_c^*(\omega)\sigma_1 x_a R - \frac{1}{2}\sigma_1\sigma_3 P_d(\omega) f - \varphi P_c^*(\omega)\sigma_1 R\right).
\end{aligned} \tag{5.3}$$

Then $d\mathrm{B}_0 = \Omega_0$ *and* $d\mathrm{B} = \Omega$.

Proof. $d\mathrm{B}_0 = \Omega_0$ follows from the definition of exterior differential. Since $|\varphi| \le C(|z| + \|f\|_{H^{-K,-S}})^2$ and since $P_d(\omega)f = (P_d(\omega) - P_d(\omega_0))f$ is a 0 of order 2 at $\omega = \omega_0$ and $f = 0$, to get $d\Gamma = 0$ at $R = 0$ and $\omega = \omega_0$ it is enough to show

$$d\left(\langle x_a\sigma_1 R|\xi_j\rangle dz_j + \langle x_a R|\xi_j\rangle d\overline{z}_j + \langle P_c^*(\omega)\sigma_1 x_a R|f'\rangle\right) = 0 \text{ at } R = 0.$$

But this is true. Formally $d\left(d\langle x_a\sigma_1 R|R\rangle - \partial_\omega\langle x_a\sigma_1 R|R\rangle d\omega\right) = 0$ at $R = 0$.
We prove the formula for B. Set $\widetilde{B} := \frac{1}{2}\langle \sigma_1\sigma_3 U|$. Notice that $d\widetilde{B} = \Omega$. We get

$$\begin{aligned}
\widetilde{B} = &\frac{1}{2}\langle \tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_1\sigma_3\Phi| + \frac{1}{2}\langle \tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_1\sigma_3 P_c(\omega) f| \\
&+ \frac{1}{2}z_j\langle \tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_1\sigma_3\xi_j| - \frac{1}{2}\overline{z}_j\langle \tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_3\xi_j|,
\end{aligned} \tag{5.4}$$

using (2.14). By Lemmas 2.3 and 2.4, the sum of the last three terms equals

$$\begin{aligned}
&\frac{1}{2}z_j d\overline{z}_j - \frac{1}{2}\overline{z}_j dz_j + \frac{1}{2}\langle \sigma_1\sigma_3 f|P_c(\omega) f'\rangle + \frac{1}{2}\langle \sigma_1\sigma_3 R|\partial_\omega R\rangle d\omega \\
&- \mathrm{i}Q(R)d\vartheta - \frac{\mathrm{i}}{4}\langle \sigma_1 R|x_a R\rangle dv_a - \mathrm{i}\left(\Pi_a(R) - \frac{v_a}{2}Q(R)\right)dD_a.
\end{aligned} \tag{5.5}$$

Applying $\frac{1}{2}\tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_1\sigma_3$ to decomposition (2.10) for $X=\Phi$, we get

$$\begin{aligned}&\frac{1}{2}\langle\tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_1\sigma_3\Phi| = -\frac{q}{q'}\langle\tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_3\partial_\omega\Phi| - \frac{1}{2}\langle\tau_D e^{-\mathrm{i}\sigma_3\Theta}P_c(\mathcal{H}_\omega^*)\sigma_3\Phi| \\ &-\frac{1}{2}\langle\sigma_3\Phi|\xi_j\rangle\langle\tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_3\xi_j| + \frac{1}{2}\langle\sigma_3\Phi|\xi_j\rangle\langle\tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_1\sigma_3\xi_j|.\end{aligned} \tag{5.6}$$

By (2.19) we have

$$\begin{aligned}&-\frac{q}{q'}\langle\tau_D e^{-\mathrm{i}\sigma_3\Theta}\sigma_3\partial_\omega\Phi| = \frac{q}{q'}\langle R|\sigma_3\partial_\omega^2\Phi\rangle\, d\omega - \mathrm{i}\,\frac{q}{q'}\left(q' + \langle R|\partial_\omega\Phi\rangle\right) d\vartheta \\ &+\frac{q}{q'}\left(\frac{\mathrm{i}v_a}{2}q' + \langle(\partial_a + \frac{\mathrm{i}}{2}\sigma_3 v_a)R|\sigma_3\partial_\omega\Phi\rangle\right) dD_a - \mathrm{i}\frac{q}{q'}\langle x_a R, \partial_\omega\Phi\rangle dv_a.\end{aligned} \tag{5.7}$$

Set $\psi(U) := \frac{1}{2}\langle\sigma_3\Phi|R\rangle$. We have

$$d\psi = \frac{1}{2}\langle\sigma_3\Phi|\partial_\omega R\rangle d\omega + \frac{1}{2}\langle\sigma_3\Phi|\xi_j\rangle\,(dz_j - d\overline{z}_j) + \frac{1}{2}\langle P_c^*(\omega)\sigma_3\Phi|f'\rangle. \tag{5.8}$$

We have $\nabla_f\psi = \frac{1}{2}P_c^*(\omega_0)P_c^*(\omega)\sigma_3\Phi$. We will use the notation $\partial_j := \partial_{z_j}$ and $\partial_{\overline{j}} := \partial_{\overline{z}_j}$. The sum of the last three terms in the rhs of (5.6) equals

$$\begin{aligned}&-\partial_j\psi dz_j - \partial_{\overline{j}}\psi d\overline{z}_j - \langle\nabla_f\psi|f'\rangle - \frac{1}{2}\langle P_{N_g^\perp(\mathcal{H}_\omega^*)}\Phi|\sigma_3\partial_\omega R\rangle d\omega \\ &-\frac{\mathrm{i}}{2}\langle P_{N_g(\mathcal{H}_\omega)}\Phi|\sigma_3 R\rangle\, d\vartheta - \frac{\mathrm{i}}{4}\langle P_{N_g(\mathcal{H}_\omega)}\Phi|x_a R\rangle dv_a \\ &+\frac{1}{2}\langle P_{N_g(\mathcal{H}_\omega)}\Phi|\sigma_3(\partial_a + \mathrm{i}\sigma_3\frac{v_a}{2})R\rangle dD_a.\end{aligned} \tag{5.9}$$

Then we get

$$\begin{aligned}&\widetilde{B} = \frac{z_j d\overline{z}_j - \overline{z}_j dz_j}{2} + \frac{1}{2}\langle f|\sigma_3\sigma_1 P_c(\omega)f'\rangle - \partial_j\psi dz_j \\ &-\partial_{\overline{j}}\psi d\overline{z}_j - \langle\nabla_f\psi|f'\rangle - \mathrm{i}\left(Q + \left\langle\frac{q}{q'}\partial_\omega\Phi - \frac{1}{2}\sigma_3 P_{N_g(\mathcal{H}_\omega)}\Phi|R\right\rangle\right) d\vartheta \\ &+\left(\frac{1}{2}\langle\sigma_1 R|\sigma_3\partial_\omega R\rangle - \left\langle\frac{q}{q'}\partial_\omega\Phi + \frac{1}{2}P_{N_g^\perp(\mathcal{H}_\omega^*)}\Phi|\sigma_3\partial_\omega R\right\rangle\right) d\omega \\ &+\left(\mathrm{i}\Pi_a + \left\langle\frac{q}{q'}\partial_\omega\Phi - \frac{1}{2}P_{N_g(\mathcal{H}_\omega)}\Phi|\sigma_3(\partial_a + \mathrm{i}\sigma_3\frac{v_a}{2})R\right\rangle\right) dD_a \\ &-\frac{\mathrm{i}}{2}\left(\frac{1}{2}\langle\sigma_1 R|x_a R\rangle + \left\langle\frac{q}{q'}\partial_\omega\Phi - \frac{1}{2}P_{N_g(\mathcal{H}_\omega)}\Phi|x_a R\right\rangle\right) dv_a.\end{aligned} \tag{5.10}$$

By (2.10) have the following two equalities:

$$\begin{aligned}&\frac{1}{2}P_{N_g(\mathcal{H}_\omega)}\Phi = \frac{\langle\Phi|\Phi\rangle}{2q'}\partial_\omega\Phi = \frac{q}{q'}\partial_\omega\Phi, \\ &\Phi = \frac{q}{q'}\partial_\omega\Phi + \frac{1}{2}P_{N_g^\perp(\mathcal{H}_\omega^*)}\Phi.\end{aligned} \tag{5.11}$$

By (5.11) we have various cancelations in (5.10) yielding

$$\begin{aligned}\widetilde{B} &= \frac{z_j d\overline{z}_j - \overline{z}_j dz_j}{2} + \frac{1}{2}\langle f|\sigma_3\sigma_1 P_c(\omega) f'\rangle - d\psi - \mathrm{i}Q d\vartheta - \mathrm{i}\Pi_a dD_a \\ &+ \frac{1}{2}\langle \sigma_1\sigma_3 R|\partial_\omega R\rangle d\omega - \frac{\mathrm{i}}{4}\langle \sigma_1 R|x_a R\rangle dv_a.\end{aligned} \tag{5.12}$$

We have $\Omega = d(\widetilde{B} + d\psi) = d\mathrm{B}$ if we define

$$\begin{aligned}\mathrm{B} &:= \mathrm{i}\vartheta dQ + \mathrm{i}D_a d\Pi_a - \frac{\overline{z}_j dz_j - z_j d\overline{z}_j}{2} + \frac{1}{2}\langle f|\sigma_3\sigma_1 P_c(\omega) f'\rangle \\ &+ \frac{1}{2}\left(\langle \sigma_1\sigma_3 R|\partial_\omega R\rangle + \mathrm{i}v_a\langle \sigma_1 R|x_a\partial_\omega R\rangle\right) d\omega \\ &+ \frac{\mathrm{i}}{2} v_a\langle x_a\sigma_1 R|\xi_j\rangle dz_j + \frac{\mathrm{i}}{2} v_a\langle x_a R|\xi_j\rangle dz_j + \frac{\mathrm{i}}{2} v_a\langle x_a\sigma_1 R|P_c(\omega) f'\rangle.\end{aligned} \tag{5.13}$$

Finally, this B satisfies (5.2)–(5.3) if we consider the formula

$$(q' + \partial_\omega Q(R))d\omega = dQ - \partial_j Q(R) dz_j - \partial_{\overline{j}} Q(R) d\overline{z}_j - \langle \nabla_f Q(R)|f'\rangle \tag{5.14}$$

where $\nabla_f Q(R) = P_c^*(\omega_0) P_c^*(\omega)\sigma_1 R$. □

In general, given a function F, the following formula defines $\nabla_f F$:

$$\begin{aligned}dF &= \partial_Q F dQ + \partial_{\Pi_a} F d\Pi_a + \partial_\vartheta F dQ + \partial_{D_a} F dD_a \\ &+ \partial_j F dz_j + \partial_{\overline{j}} F dz_{\overline{j}} + \langle \nabla_f F|f'\rangle,\end{aligned}$$

where $P_c^*(\omega_0)\nabla_f F = \nabla_f F$. We have the following result.

Lemma 5.2. *Let us denote by $\mathcal{U}_\Sigma$ the subset of Σ defined by the inequalities $|z| \le \varepsilon_0$, $\|f\|_{H^{-K,-S}} \le \varepsilon_0$, $|\varrho(f)| \le \varepsilon_0$ (here recall $\varrho(f) := (Q(f), \Pi(f))$), $|\omega - \omega_0| \le \varepsilon_0$ and $|v| \le \varepsilon_0$. Then there exists a number $\varepsilon_0 > 0$ such that there exists a unique vectorfield $\mathcal{X}^t : U_\Sigma \to L^2$ which solves $i_{\mathcal{X}^t}\Omega_t = -\Gamma$, where $\Omega_t := \Omega_0 + t(\Omega - \Omega_0)$.*

Remark 5.3. In [Cu1] the existence of $\mathcal{X}^t$ is elementary and standard, due to the fact that Ω_0 and Ω are differential forms in L^2 very close to each other. This is not true any more in our setting, where $\Omega_0 : \Sigma \to B^2(L^2, \mathbb{C})$ and were we cannot assume that Ω_0 and Ω are close in Σ. This for the simple reason that while the spaces Σ_n are invariant for our NLS (1.1), see Lemma 7.1 below, for $n > 0$ the Σ_n norms of the solutions grow in time. One of the main differences between [Cu1] and this paper lies here.

Proof. The proof ends after Lemma 5.7. We are considering $i_{\mathcal{X}^t}\Omega_0 + t\, i_{\mathcal{X}^t} d\Gamma = -\Gamma$ where

$$\begin{aligned}i_X\Omega_0 &= \mathrm{i}(X)_\vartheta dQ + \mathrm{i}(X)_{D_a} d\Pi_a - \mathrm{i}(X)_Q d\vartheta - \mathrm{i}(X)_{\Pi_a} dD_a \\ &- (X)_{\overline{j}} dz_j + (X)_j dz_{\overline{j}} + \langle \sigma_1\sigma_3 (X)_f|f'\rangle,\end{aligned} \tag{5.15}$$

$(X)_\vartheta$ the ϑ–th component of X, etc. Set

$$\begin{aligned}
\gamma_{1a} &:= \langle x_a\sigma_1 R|\xi_j\rangle dz_j + \langle x_a R|\xi_j\rangle d\overline{z}_j + \langle P_c^*(\omega)\sigma_1 x_a R|f'\rangle \ , \\
W &:= \varphi dQ - \varphi\partial_j Q(R)dz_j - \varphi\partial_{\overline{j}} Q(R)d\overline{z}_j \\
&\quad - \varphi\langle P_c^*(\omega)\sigma_1 R|f'\rangle - 2^{-1}\langle \sigma_1\sigma_3 P_d(\omega)f|f'\rangle.
\end{aligned} \tag{5.16}$$

We have $\Gamma = \frac{\mathrm{i}}{2}v_a\gamma_{1a} + W$ and

$$d\Gamma = \frac{\mathrm{i}}{2}dv_a\wedge\gamma_{1a} + \frac{\mathrm{i}}{2}v_a d\gamma_{1a} + dW. \tag{5.17}$$

Lemma 5.4. *Using definitions* (5.21)–(5.24) *below, we have:*

$$\gamma_{1a} = \langle P_c^*(\omega)\sigma_1 x_a f|f'\rangle + \widetilde{\gamma}_{1a}; \tag{5.18}$$

$$\begin{aligned}
dv_a &= -\frac{2}{Q}d\Pi_a + 2\left[(\Pi_a - \Pi_a(R))\, Q^{-2} + \frac{Q^{-1}\partial_\omega\Pi_a(R)}{q' + \partial_\omega Q(R)}\right] dQ \\
&+ 2Q^{-1}\left\langle \mathrm{i}P_c^*(\omega)\sigma_1\sigma_3\partial_a f - \frac{\partial_\omega\Pi_a(R)P_c^*(\omega)\sigma_1 f}{q' + \partial_\omega Q(R)}|f'\right\rangle + \widetilde{dv_a};
\end{aligned} \tag{5.19}$$

$$d\gamma_{1a} = \widehat{\gamma}_{1a}\wedge(dQ - \langle P_c^*(\omega)\sigma_1 f|f'\rangle) + \widetilde{d\gamma}_{1a}. \tag{5.20}$$

In the above formulas we have

$$\widetilde{\gamma}_{1a} : \ = \langle x_a\sigma_1 R|\xi_j\rangle dz_j + \langle x_a R|\xi_j\rangle d\overline{z}_j + \langle P_c^*(\omega)x_a(z\cdot\sigma_1\xi + \overline{z}\cdot\xi|f'\rangle\,; \tag{5.21}$$

$$\begin{aligned}
\frac{Q}{2}\widetilde{dv_a} &= \left(Q^{-1}\partial_j\Pi_a(R) - \frac{\partial_\omega\Pi_a(R)\partial_j Q(R)}{q' + \partial_\omega Q(R)}\right) dz^j \\
&+ \left(Q^{-1}\partial_{\overline{j}}\Pi_a(R) - \frac{\partial_\omega\Pi_a(R)\partial_{\overline{j}} Q(R)}{q' + \partial_\omega Q(R)}\right) d\overline{z}^j - \\
&\langle \mathrm{i}P_c^*(\omega)\sigma_3\partial_a(z\cdot\sigma_1\xi + \overline{z}\cdot\xi) + \frac{\partial_\omega\Pi_a(R)P_c^*(\omega)(z\cdot\sigma_1\xi + \overline{z}\cdot\xi)}{q' + \partial_\omega Q(R)}|f'\rangle;
\end{aligned} \tag{5.22}$$

$$\widehat{\gamma}_{1a} := \frac{\partial_\omega\langle x_a\sigma_1 R|\xi_j\rangle dz_j + \partial_\omega\langle x_a R|\xi_j\rangle d\overline{z}_j - \langle\partial_\omega P_d^*(\omega)\sigma_1 x_a R|f'\rangle}{q' + \partial_\omega Q(R)}; \tag{5.23}$$

$$\widetilde{d\gamma}_{1a} = (\partial_k Q(R)dz_k + \partial_{\overline{k}} Q(R)d\overline{z}_k + \langle P_c^*(\omega)(z\cdot\sigma_1\xi + \overline{z}\cdot\xi)|f'\rangle)\wedge\widehat{\gamma}_{1a}. \tag{5.24}$$

Proof. By (4.5) we get

$$\begin{aligned}
dv_a &= -\frac{2}{Q}d\Pi_a + 2(\Pi_a - \Pi_a(R))Q^{-2}dQ + 2Q^{-1}\partial_j\Pi_a(R)dz^j \\
&+ 2Q^{-1}\partial_{\overline{j}}\Pi_a(R)d\overline{z}^j + 2Q^{-1}\langle\nabla_f\Pi_a(R)|f'\rangle + 2Q^{-1}\partial_\omega\Pi_a(R)d\omega,
\end{aligned}$$

where $\nabla_f\Pi_a(R) = \mathrm{i}P_c^*(\omega_0)P_c^*(\omega)\sigma_1\sigma_3\partial_a R$. Then by (5.14) we get

$$
\begin{aligned}
dv_a &= -\frac{2}{Q}d\Pi_a + 2\left[(\Pi_a - \Pi_a(R))\,Q^{-2} + \frac{Q^{-1}\partial_\omega \Pi_a(R)}{q' + \partial_\omega Q(R)}\right] dQ \\
&+ 2Q^{-1}\left[Q^{-1}\partial_j \Pi_a(R) - \frac{\partial_\omega \Pi_a(R)\partial_j Q(R)}{q' + \partial_\omega Q(R)}\right] dz^j \\
&+ 2Q^{-1}\left[Q^{-1}\partial_{\overline{j}} \Pi_a(R) - \frac{\partial_\omega \Pi_a(R)\partial_{\overline{j}} Q(R)}{q' + \partial_\omega Q(R)}\right] d\overline{z}^j \\
&+ 2Q^{-1}\left\langle \mathrm{i}P_c^*(\omega)\sigma_1\sigma_3\partial_a R - \frac{\partial_\omega \Pi_a(R) P_c^*(\omega)\sigma_1 R}{q' + \partial_\omega Q(R)}|f'\right\rangle .
\end{aligned}
$$

In particular we get (5.19) and (5.22). (5.21) is immediate from (5.16). (5.23)–(5.24) follow by (5.14) from

$$
\begin{aligned}
&d\gamma_{1a} = (\partial_\omega \langle x_a \sigma_1 R|\xi_j\rangle dz_j + \partial_\omega \langle x_a R|\xi_j\rangle d\overline{z}_j - \langle \partial_\omega P_d^*(\omega)\sigma_1 x_a R|f'\rangle) \\
&\wedge \frac{dQ - \partial_k Q(R) dz_k - \partial_{\overline{k}} Q(R) d\overline{z}_k - \langle P_c^*(\omega)\sigma_1 R|f'\rangle}{q' + \partial_\omega Q(R)} \\
&= \widehat{\gamma}_{1a} \wedge ((dQ - \langle P_c^*(\omega)\sigma_1 R|f'\rangle) + (\partial_k Q(R) dz_k - \partial_{\overline{k}} Q(R) d\overline{z}_k)).
\end{aligned}
$$

$\square$

Set $H_c^{K,S} = P_c(\omega_0)H^{K,S}$ and denote

$$
\mathcal{P}^{K,S} = \mathbb{R}^6 \times \mathbb{C}^m \times H_c^{K,S}, \quad \mathcal{P} = \mathcal{P}^{0,0}. \tag{5.25}
$$

The following lemma is straightforward.

Lemma 5.5. *For $\varpi_1 = \widetilde{dv_a}$, $\widetilde{\gamma}_{1a}$, $\widehat{\gamma}_{1a}$, we have $\varpi_1 \in C^\infty(\mathcal{U}_\Sigma, B(\mathcal{P}^{-K,-S}, \mathbb{C}))$ with for fixed C*

$$
\|\varpi_1\|_{B(\mathcal{P}^{-K,-S},\mathbb{C})} \le C(|z| + \|f\|_{H^{-K,-S}}). \tag{5.26}
$$

For $\varpi_2 = \widetilde{d\gamma_{1a}}$ we have $\varpi_2 \in C^\infty(\mathcal{U}_\Sigma, B^2(\mathcal{P}^{-K,-S}, \mathbb{C}))$ with for fixed C

$$
\|\varpi_2\|_{B^2(\mathcal{P}^{-K,-S},\mathbb{C})} \le C(|z| + \|f\|_{H^{-K,-S}})^2. \tag{5.27}
$$

Furthermore, consider the contraction operator $X \to i_X \varpi_2$ and define $Y(X)$ by $i_{Y(X)}\Omega_0 = i_X \varpi_2$. Then we have an equality of the form, summing on repeated index, with finite sums,

$$
\begin{aligned}
&(Y(X))_{\underline{j}} = R_k^{(\underline{j})}(X)\widetilde{\lambda}_k^{(\underline{j})} \text{ for } \underline{j} = j, \overline{j} \\
&(Y(X))_f = R_k(X)\Lambda_k
\end{aligned} \tag{5.28}
$$

with $R_k^{(\underline{j})}, R_k \in C^\infty(\mathcal{U}_\Sigma, B(\mathcal{P}^{-K,-S}, \mathbb{C}))$ satisfying

$$
\|R_k^{(\underline{j})}\|_{B(\mathcal{P}^{-K,-S},\mathbb{C})} + \|R_k\|_{B(\mathcal{P}^{-K,-S},\mathbb{C})} \le C(|z| + \|f\|_{H^{-K,-S}}), \tag{5.29}
$$

with $\widetilde{\lambda}_k^{(j)} \in C^\infty(\mathcal{U}_\Sigma, \mathbb{C})$ *and* $\Lambda_k \in C^\infty(\mathcal{U}_\Sigma, H^{K,S})$ *satisfying estimates*

$$|\widetilde{\lambda}_k^{(j)}| + \|\Lambda_k\|_{H^{K,S}} \le C(|z| + \|f\|_{H^{-K,-S}}). \tag{5.30}$$

For $E = \varpi_1,\ \varpi_2,\ R_k^{(j)},\ R_k,\ \widetilde{\lambda}_k^{(j)}$ *and* Λ_k*, we have* $E = E(Q, \Pi, z, f, \varrho(f))$*, where* $E(Q, \Pi, z, f, \varrho)$ *is smooth w.r.t.* (Q, Π) *and* $\varrho \in \mathbb{R}^4$*,* $z \in \mathbb{C}^m$ *and* $f \in H^{-K,-S}$*.* $\square$

Lemma 5.6. *Set*

$$\begin{aligned}
&G_1 := \frac{1}{2}\langle \sigma_1\sigma_3 R|\partial_\omega R\rangle\,, \quad G_{2a} := \frac{\mathrm{i}}{2}\langle x_a\sigma_1 R|\partial_\omega R\rangle\ ,\\
&\widehat{W} := (\partial_j G_1 + v_a\partial_j G_{2a})dz_j + (\partial_{\overline{j}} G_1 + v_a\partial_{\overline{j}} G_{2a})d\overline{z}_j +\\
&\langle \nabla_f G_1 + v_a\nabla_f G_{2a} + 2^{-1}\sigma_1\sigma_3\partial_\omega P_d(\omega)f|f'\rangle + G_{2a}\widetilde{dv_a}\,,\\
&\widetilde{dW} := -\widehat{W}\wedge \frac{\partial_j Q(R)dz_j + \partial_{\overline{j}} Q(R)d\overline{z}_j + \langle P_c^*(\omega)(z\cdot\sigma_1\xi + \overline{z}\cdot\xi)|f'\rangle}{q' + \partial_\omega Q(R)}\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
&dW = \widehat{W}\wedge\frac{dQ - \langle P_c^*(\omega)\sigma_1 f|f'\rangle}{q' + \partial_\omega Q(R)} + \widetilde{dW} +\\
&G_{2a}(dv_a - \widetilde{dv_a})\wedge \frac{dQ - \partial_j Q(R)dz_j - \partial_{\overline{j}} Q(R)d\overline{z}_j - \langle P_c^*(\omega)\sigma_1 R|f'\rangle}{q' + \partial_\omega Q(R)},
\end{aligned} \tag{5.31}$$

where, for $\varpi_1 = \widehat{W}$ *and* $\varpi_2 = \widetilde{dW}$*, the conclusions of Lemma 5.5 are satisfied.*

Proof. Substituting (5.14) and using (5.3) we get

$$W = (G_1 + v_a G_{2a})d\omega - 2^{-1}\langle\sigma_1\sigma_3 P_d(\omega)f|f'\rangle. \tag{5.32}$$

By (5.14) we obtain the following formula, from which we get (5.31):

$$\begin{aligned}
dW &= \big(dG_1 + v_a dG_{2a} + 2^{-1}\langle\sigma_1\sigma_3\partial_\omega P_d(\omega)f|f'\rangle + G_{2a}dv_a\big)\wedge d\omega\\
&= \Big(\widehat{W} + G_{2a}(dv_a - \widetilde{dv_a})\Big)\wedge d\omega.
\end{aligned}$$

Then substitute (5.14). $\square$

We reframe the equation for $\mathcal{X}^t$.

Lemma 5.7. *For* Y *defined by* $i_Y\Omega_0 = -\Gamma$*. Then equation* $i_{\mathcal{X}^t}\Omega_t = -\Gamma$*, is equivalent to*

$$(1 + t\mathcal{K})\mathcal{X}^t = Y \tag{5.33}$$

where the operator $\mathcal{K}$ *satisfies the following properties.*

(1) For the component $(\mathcal{K}X)_f$ the following facts hold.

$$(\mathcal{K}X)_f = A_a(X)\partial_a f + \sigma_3(B_a(X)x_a + C(X))f + D_i(X)\Psi_i \tag{5.34}$$

where the last is a finite sum with $\Psi_i \in C^\infty(\mathcal{U}_\Sigma, H^{K,S})$ with

$$\|\Psi_i\|_{H^{K,S}} \le C(|z| + \|f\|_{H^{-K,-S}}). \tag{5.35}$$

For $L = A_a, B_a, C, D_i$, we have $L \in C^0(\mathcal{U}_\Sigma, B(\mathcal{P}, \mathbb{C}))$, see (5.25)*, with*

$$L(X) = \langle L_{1b}\sigma_1\sigma_3\partial_b f + L_{2b}\sigma_1 x_b f + L_3\sigma_1 f | (X)_f\rangle + \widetilde{L}(X) \tag{5.36}$$

with $\widetilde{L} \in C^\infty(\mathcal{U}_\Sigma, B(\mathcal{P}^{-K,-S}, \mathbb{C}))$, $\|\widetilde{L}\|_{B(\mathcal{P}^{-K,-S},\mathbb{C})} \le C(|z| + \|f\|_{H^{-K,-S}})$ and where L_{1b}, L_{2b}, L_3 are in $C^\infty(\mathcal{U}_\Sigma, \mathbb{C})$.

(2) The z_j–th component $(\mathcal{K}X)_j$ is of the form (5.36) *with the estimates*

$$|L_{1b}| + |L_{2b}| + |L_3| + \|\widetilde{L}\|_{B(\mathcal{P}^{-K,-S},\mathbb{C})} \le C(|z| + \|f\|_{H^{-K,-S}}). \tag{5.37}$$

(3) For G any of the above L_{1b}, L_{2b}, L_3, $\widetilde{L}$, we have $G = G(Q, \Pi, z, f, \varrho(f))$, where $G(Q, \Pi, z, f, \varrho)$ is smooth w.r.t. (Q, Π) and $\varrho \in \mathbb{R}^4$, $z \in \mathbb{C}^m$ and $f \in H^{-K,-S}$.

Proof. We have $i_{\frac{\partial}{\partial Q}}\Omega_0 = -\mathrm{i}d\vartheta$ by (5.1) and $i_{\frac{\partial}{\partial Q}}\Omega = -\mathrm{i}d\vartheta$ by Lemmas 4.1 and 4.4. Similarly $i_{\frac{\partial}{\partial \Pi_a}}\Omega_0 = i_{\frac{\partial}{\partial \Pi_a}}\Omega = -\mathrm{i}dD_a$. So in particular $i_{\frac{\partial}{\partial Q}}d\Gamma = i_{\frac{\partial}{\partial D_a}}d\Gamma = 0$. Then $(\Gamma)_\vartheta = (\Gamma)_{D_a} = 0$ implies $(\mathcal{X}^t)_Q = (\mathcal{X}^t)_{\Pi_a} = 0$. Then

$$\begin{aligned}
i_{\mathcal{X}^t}d\Gamma &= \frac{\mathrm{i}}{2}\left(dv_a(\mathcal{X}^t)\gamma_{1a} - \gamma_{1a}(\mathcal{X}^t)dv_a + v_a i_{\mathcal{X}^t}d\gamma_{1a}\right) + i_{\mathcal{X}^t}dW \\
&= \frac{\mathrm{i}}{2}dv_a(\mathcal{X}^t)\left(\langle P_c^*(\omega)\sigma_1 x_a f|f'\rangle + \widetilde{\gamma}_{1a}\right) - \frac{\mathrm{i}}{2}\gamma_{1a}(\mathcal{X}^t)\left(\widetilde{dv_a} + dv_a - \widetilde{dv_a}\right) \\
&+ \frac{\mathrm{i}}{2}v_a\widehat{\gamma}_{1a}(\mathcal{X}^t)(dQ - \langle P_c^*(\omega)\sigma_1 f|f'\rangle) + \frac{\mathrm{i}}{2}v_a\langle P_c^*(\omega)\sigma_1 f|(\mathcal{X}^t)_f\rangle\widehat{\gamma}_{1a} \\
&+ \widehat{W}(\mathcal{X}^t)\frac{dQ - \langle P_c^*(\omega)\sigma_1 f|f'\rangle}{q' + \partial_\omega Q(R)} + \frac{\langle P_c^*(\omega)\sigma_1 f|(\mathcal{X}^t)_f\rangle}{q' + \partial_\omega Q(R)}\widehat{W} + i_{\mathcal{X}^t}\widetilde{dW} \\
&+ G_{2a}(dv_a(\mathcal{X}^t) - \widetilde{dv_a}(\mathcal{X}^t))\frac{dQ - \partial_j Q(R)dz_j - \partial_{\overline{j}}Q(R)d\overline{z}_j - \langle P_c^*(\omega)\sigma_1 R|f'\rangle}{q' + \partial_\omega Q(R)} \\
&+ G_{2a}\frac{\partial_j Q(R)(\mathcal{X}^t)_j + \partial_{\overline{j}}Q(R)(\mathcal{X}^t)_{\overline{j}} + \langle P_c^*(\omega)\sigma_1 R|(\mathcal{X}^t)_f\rangle}{q' + \partial_\omega Q(R)}(dv_a - \widetilde{dv_a}).
\end{aligned}$$

So for $\widetilde{Q}$ and $\widetilde{\Pi}_a$ two 1–forms irrelevant in the sequel (since we are not interested about $(\mathcal{X}^t)_\vartheta$ and $(\mathcal{X}^t)_{D_a}$), after a tedious but elementary computation we have

$$
\begin{aligned}
&i_{\mathcal{X}^t} d\Gamma = \widetilde{Q}(\mathcal{X}^t) dQ + \widetilde{\Pi}_a(\mathcal{X}^t) d\Pi_a + \frac{\mathrm{i}}{2} dv_a(\mathcal{X}^t) \widetilde{\gamma}_{1a} - \frac{\mathrm{i}}{2} \gamma_{1a}(\mathcal{X}^t) \widetilde{dv_a} \\
&+ \frac{\mathrm{i}}{2} v_a i_{\mathcal{X}^t} \widetilde{d\gamma}_{1a} + i_{\mathcal{X}^t} \widetilde{dW} + \langle P_c^*(\omega)\sigma_1 f | (\mathcal{X}^t)_f \rangle \left(\frac{\mathrm{i}}{2} v_a \widehat{\gamma}_{1a} + \frac{\widehat{W}}{q' + \partial_\omega Q(R)} \right) \\
&- (q' + \partial_\omega Q(R))^{-1} G_{2a} (dv_a(\mathcal{X}^t) - \widetilde{dv_a}(\mathcal{X}^t)) (\partial_j Q(R) dz_j + \partial_{\overline{j}} Q(R) d\overline{z}_j) \\
&+ \widehat{\Gamma}_3(\mathcal{X}^t) \langle P_c^*(\omega)\sigma_1 f | f' \rangle + \widehat{\Gamma}_{2a}(\mathcal{X}^t) \langle P_c^*(\omega)\sigma_1 x_a f | f' \rangle \\
&+ \widehat{\Gamma}_{1a}(\mathcal{X}^t) \langle P_c^*(\omega)\sigma_1 \sigma_3 \partial_a f | f' \rangle ,
\end{aligned}
\tag{5.38}
$$

with by (5.19)

$$
\begin{aligned}
&\widehat{\Gamma}_3 := \frac{-\widehat{W}}{q' + \partial_\omega Q(R)} - \frac{\mathrm{i}}{2} v_a \widehat{\gamma}_{1a} + \mathrm{i} Q^{-1} \frac{\partial_\omega \Pi_a(R)}{q' + \partial_\omega Q(R)} \gamma_{1a} \\
&- \frac{2G_{2a}}{Q(q' + \partial_\omega(R))} \left\langle \mathrm{i} P_c^*(\omega)\sigma_1\sigma_3 \partial_a f - \frac{\partial_\omega \Pi_a(R) P_c^*(\omega)\sigma_1 f}{q' + \partial_\omega Q(R)} | f' \right\rangle \\
&- 2G_{2a} \partial_\omega \Pi_a(R) \frac{\partial_k Q(R) dz_k + \partial_{\overline{k}} Q(R) d\overline{z}_k + \langle P_c^*(\omega)\sigma_1 R | f' \rangle}{Q(q' + \partial_\omega Q(R))^2} \\
&\widehat{\Gamma}_{2a} := \frac{\mathrm{i}}{2} dv_a \\
&\widehat{\Gamma}_{1a} := \frac{\gamma_{1a}}{Q} + 2G_{2a} \frac{\partial_k Q(R) dz_k + \partial_{\overline{k}} Q(R) d\overline{z}_k + \langle P_c^*(\omega)\sigma_1 R | f' \rangle}{Q(q' + \partial_\omega Q(R))} .
\end{aligned}
\tag{5.39}
$$

The operator $\mathcal{K}$ is defined by $i_{\mathcal{K}X}\Omega_0 = i_X d\Gamma$ for any X. By (5.15) this implies that

$$
\begin{aligned}
(\mathcal{K}X)_f &= \widehat{\Gamma}_3(X) P_c(\omega_0) P_c(\omega) \sigma_3 f + \widehat{\Gamma}_{2a}(X) P_c(\omega_0) P_c(\omega) \sigma_3 x_a f + (\widehat{\mathcal{R}}(X))_f \\
&+ \widehat{\Gamma}_{1a}(X) P_c(\omega_0) P_c(\omega) \partial_a f + \langle P_c^*(\omega)\sigma_1 f | (X)_f \rangle \sigma_3 \sigma_1 \left(\frac{\mathrm{i}}{2} v_a (\widehat{\gamma}_{1a})_f + (\widehat{W})_f \right) ,
\end{aligned}
\tag{5.40}
$$

where $i_{\widehat{\mathcal{R}}(X)}\Omega_0 := \frac{\mathrm{i}}{2} v_a i_X \widetilde{d\gamma}_{1a} + i_X \widetilde{dW}$ and where $(\widehat{\gamma}_{1a})_f$ resp. $(\widehat{W})_f$ are the analogues of $(\Gamma)_f$ of the expansion of Γ under (5.2). They are like the Ψ_i of the statement by Lemma 5.5 resp. 5.6. By Lemma 5.5 we have $(\widehat{\mathcal{R}}(X))_f = R_k(X)\Lambda_k$, a sum of the form (5.28) which satisfies (5.29)–(5.30). Claim (1) follows from (5.40) after expansions like $P_c(\omega_0)P_c(\omega)\sigma_3 f = \sigma_3 f + (1 - P_c(\omega_0)P_c(\omega))\sigma_3 f$, where the second term on the right is like a Ψ_i.

The terms in (5.38) contributing to $(\mathcal{K}X)_j$ are the last two in the first line and those in the second and third lines. Then Claim (2) follows from Lemma 5.5. Claim (3) follows from Lemmas 5.1 and 5.5. □

We can apply Fredholm alternative to prove existence and uniqueness of a solution to (5.33).We check that $(1 + t\mathcal{T})X = 0$ admits no solution, with $\mathcal{T}$ the adjoint of $\mathcal{K}$ with respect to Ω_0. $\mathcal{T}$ is like $\mathcal{K}$ and satisfies (5.34)–(5.37). We

show that $X=-t\mathcal{T}X$ does not have nontrivial solutions for $|t|\le 3$ for the ε_0, see the statement of Lemma 5.2, sufficiently small. We prove this by showing that there is a fixed constant κ such that the following holds:

$$\begin{aligned}
&(\mathcal{T}^n X)_f = A_a^{(n)}(X)\partial_a f + \sigma_3[B_a^{(n)}(X)x_a + C^{(n)}(X)]f + D_i^{(n)}(X)\Psi_i,\\
&(\mathcal{T}^n X)_{\underline{j}} = Z_{\underline{j}}^{(n)}(X),\\
&|L^{(n)}(X)| \le \kappa^{n-1}(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|)^{n-1}|L(X)|,
\end{aligned} \tag{5.41}$$

for $L^{(n)} = A_a^{(n)}$, $B_a^{(n)}$, $C^{(n)}$, $D_1^{(n)}$, $Z_{\underline{j}}^{(n)}$. This will imply that the solutions of $X=-t\mathcal{T}X$ are trivial for $\kappa\varepsilon_0$ sufficiently small. We have

$$\begin{aligned}
&L^{(n+1)}(X) = \widetilde{L}(\mathcal{K}^n X)\\
&+\langle L_{1b}\sigma_1\sigma_3\partial_b f + L_{2b}\sigma_1 x_b f + L_3\sigma_1 f|(\mathcal{T}^n X)_f\rangle\,.
\end{aligned} \tag{5.42}$$

We have $|\widetilde{L}(\mathcal{K}^n X)| \le c_0\kappa^{n-1}(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|)^n|L(X)|$ for a fixed constant $c_0 = c_0(\widetilde{L})$, by induction and Lemma 5.7. Substituting (5.41), the second line in (5.42) becomes

$$\begin{aligned}
&\langle L_{1b}\sigma_1\sigma_3\partial_b f + L_{2b}\sigma_1 x_b f + L_3\sigma_1 f|A_a^{(n)}(X)\partial_a f + \cdots + D_i^{(n)}(X)\Psi_i\rangle\\
=&\delta_{ab}(L_{1b}B_a^{(n)}(X) - L_{2b}A_a^{(n)}(X))\,Q(f)\\
+&D_i^{(n)}(X)\langle L_{1b}\sigma_1\sigma_3\partial_b f + L_{2b}\sigma_1 x_b f + L_3\sigma_1 f|\Psi_i\rangle,
\end{aligned} \tag{5.43}$$

where we used

$$\begin{aligned}
&\langle\sigma_1\sigma_3\partial_a f|\partial_b f\rangle = \langle\sigma_1\partial_a f|f\rangle = \langle\sigma_1\sigma_3 x_a f|x_b f\rangle = \langle\sigma_1\sigma_3 x_a f|f\rangle\\
&=\langle\sigma_1\sigma_3 f|f\rangle = 0 \text{ and } 2\langle\sigma_1\partial_a f|x_b f\rangle = -\delta_{ab}\|f\|_{L^2}^2.
\end{aligned} \tag{5.44}$$

The absolute value of the rhs of (5.43) is by induction

$$\le c_1\kappa^{n-1}(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|)^n|L(X)|.$$

for a fixed constant $c_1 = c_1(L)$. So

$$|L^{(n+1)}(X)| \le \kappa^{n+1}(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|)^n|L(X)| \tag{5.45}$$

if the constant κ is chosen sufficiently large. The proof of Lemma 5.2 is concluded. The fact that $|L(X)| \le c_3(|z| + \|f\|_\Sigma)\|X\|_{L^2}$ does not need to be used. $\square$

The following one is another most important step in the proof. We need to show that the flow of $\mathcal{X}^t$ corresponds to the flow of a quasilinear hyperbolic symmetric system. To study this system, well posedness and regularity with respect to the initial data, we need more information on $\mathcal{X}^t$. We remark that $H^{K,S}$ has been fixed with any preassigned pair (K,S). We will take both K and S very large.

Lemma 5.8. *For $\mathcal{X}^t$ the vectorfield of Lemma 5.2, we have*

$$\begin{aligned} &(\mathcal{X}^t)_f = \mathcal{L}f + \mathcal{D} \quad , \quad (\mathcal{X}^t)_{\underline{j}} = Z_{\underline{j}} \text{ for } \underline{j} \in \{j, \overline{j}\}, \\ &\mathcal{L} := \mathcal{A}_a \partial_a + (\mathcal{B}_a x_a + \mathcal{C})\sigma_3 \end{aligned} \tag{5.46}$$

where the coefficients satisfy the following properties.

(1) $\mathcal{A}_a$ are real valued functions. $\mathcal{B}_a$ and $\mathcal{C}$ are imaginary valued. $Z_{\underline{j}}$ has values in $\mathbb{C}$ with $Z_{\overline{j}} = \overline{Z}_j$. $\mathcal{D}$ has values in $H^{K,S}$.

(2) We have $G = G(t, z, f, \varrho(f))$ for $G = \mathcal{A}_a, \mathcal{B}_a, \mathcal{C}, \mathcal{D}, Z_{\overline{j}}$, for functions $G(t, z, f, \varrho)$ smooth in t, z, in $f \in H^{-K,-S}$ and in ρ.

(3) We have

$$\begin{aligned} &|\mathcal{A}_a| \le C(K,S)(|z|^2 + \|f\|^2_{H^{-K,-S}} + |\varrho(f)|)\,, \\ &|Z| + |\mathcal{C}| + \|\mathcal{D}\|_{H^{K,S}} \\ &\le C(K,S)(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|)(|z| + \|f\|_{H^{-K,-S}})\ . \end{aligned} \tag{5.47}$$

(4) We have $\mathcal{B}_a = -\frac{\mathrm{i}}{2}v_a - \frac{\mathrm{i}}{2}tdv_a(\mathcal{X}^t)$.

Proof. Let us start assuming that $\mathcal{X}^t$ is of the form (5.46). Let X be a vector such that $\sigma_1 X = \overline{X}$. Then $\Gamma(X)$ is imaginary. We have $\overline{\Omega_t(\mathcal{X}^t, X)} = -\Omega_t(\sigma_1\overline{\mathcal{X}^t}, X)$. For $t = 1$ is straightforward and for $t = 0$ can be checked using Lemmas 2.3 and 2.4. Since also $\overline{\Omega_t(\mathcal{X}^t, X)} = -\Gamma(X) = \Omega_t(\mathcal{X}^t, X)$ we get $\sigma_1\overline{\mathcal{X}^t} = \mathcal{X}^t$. From this discussion we can conclude that Claim (1) holds if (5.46) is true.

Let Y be defined by $i_Y\Omega_0 = -\Gamma$. Then

$$\begin{aligned} &(Y)_j = -\Gamma_{\overline{j}}, \quad (Y)_{\overline{j}} = \Gamma_j \\ &(Y)_f = \sigma_3\sigma_1\Gamma_f = -\frac{\mathrm{i}}{2}\sigma_3 v_a x_a f + \varphi\sigma_3 f + \widetilde{Y}_f, \end{aligned} \tag{5.48}$$

with $(Y)_{\underline{j}}$ and $\widetilde{Y}_f$, like $G = G(t, z, f, \varrho(f))$ in the statement above, smooth in z, in $f \in H^{-K,-S}$ and in ρ and s.t., by (5.3),

$$|(Y)_{\underline{j}}| + \|\widetilde{Y}_f\|_{H^{K,S}} \le C(|z| + \|f\|_{H^{-K,-S}})(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|). \tag{5.49}$$

Our *first claim* is that the following series converge:

$$\begin{aligned} (\mathcal{X}^t)_f &= \sum_{n=0}^{\infty}(-1)^n t^n (\mathcal{K}^n Y)_f \\ &=: \widehat{A}_a(Y)\partial_a f + \sigma_3(\widehat{B}_a(Y)x_a + \widehat{C}(Y))f + \widehat{D}_i(Y)\Psi_i, \\ (\mathcal{X}^t)_{\underline{j}} &= \sum_{n=0}^{\infty}(-1)^n t^n (\mathcal{K}^n Y)_{\underline{j}} =: \widehat{Z}_{\underline{j}}(Y), \end{aligned} \tag{5.50}$$

with, for $L = A_a, B_a, C, D_i, Z_{\underline{j}}$,

$$\widehat{L}(Y) = \sum_{n=0}^{\infty} (-1)^n t^n L^{(n)}(Y), \tag{5.51}$$

for $L^{(n)} = A_a^{(n)}$, $B_a^{(n)}$, $C^{(n)}$, $D_i^{(n)}$, $Z_{\underline{j}}^{(n)}$ defined as in (5.41) but with $\mathcal{K}$ instead of $\mathcal{T}$. To prove the *first claim*, notice that by the proof of Lemma 5.7 we can conclude that there exists a fixed constant κ such that the following analogue of (5.45) holds:

$$\begin{aligned} &|L^{(n+1)}(Y)| \le \kappa^{n+1}(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|)^n |L(Y)| \le \\ &\kappa^{n+1}(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|)^n (|z|^2 + \|f\|^2_{H^{-K,-S}} + |\varrho(f)|), \end{aligned} \tag{5.52}$$

where in the second inequality we exploited (5.48)–(5.49) and Lemma 5.7 which yield

$$|L(Y)| \le C_L(|z|^2 + \|f\|^2_{H^{-K,-S}} + |\varrho(f)|). \tag{5.53}$$

In the case of $L = \mathcal{C}, Z, D_i$ we have a better estimate:

$$|L(Y)| \le C_0(|z| + \|f\|_{H^{-K,-S}})(|z| + \|f\|_{H^{-K,-S}} + |\varrho(f)|). \tag{5.54}$$

This yields the *first claim*, proves that $\mathcal{X}^t$ is of the form of (5.46) and that (5.47) holds. A *second claim* is that

$$L^{(n+1)}(Y) = \sum_{i_1 \in I_1} \varsigma_{i_1}(Y) \sum_{\sigma,\tau} \varepsilon_{i_1}(\sigma,\tau) \prod_{j=0}^{n} \ell_{\sigma(j)} \varpi_{\tau(j)}, \tag{5.55}$$

where we have what follows.

(i) The sums on σ and τ are over all maps $\sigma : \mathbb{Z}_n \to I_2$ and $\tau : \mathbb{Z}_n \to I_3$ with I_j, for $j = 1, 2, 3$, three finite sets described below. For $i_1 \in I_1$ are functions $(\sigma,\tau) \to \varepsilon_{i_1}(\sigma,\tau)$ with values in $\{0, 1\}$.

(ii) $\varsigma_{i_1}(Y)$ varies in all possible ways among $\langle \sigma_1\sigma_3\partial_a f|Y\rangle$, $\langle \sigma_1 x_a f|Y\rangle$, $\langle \sigma_1 f|Y\rangle$ and $\widetilde{L}(Y)$, for $L = A_a, B_a, C, D_i, (\mathcal{K} \quad)_{\underline{j}}$. We denote by I_1 the set of these functions.

(iii) ℓ_{i_2} varies among L_{1b}, L_{2b}, L_3, for $L = A_a, B_a, C, D_i, (\mathcal{K} \quad)_{\underline{j}}$. We denote by I_2 the set of these functions.

(iv) ϖ_{i_3} varies among 1, $\|f\|_2^2$, $\langle \sigma_1\sigma_3\partial_a f|\Psi_i\rangle$, $\langle \sigma_1 x_a f|\Psi_i\rangle$, $\langle \sigma_1 f|\Psi_i\rangle$, $\widetilde{L}(\Psi_i)$, $\widetilde{L}(\Psi_i)$, $\widetilde{L}(\partial_a f)$, $\widetilde{L}(\sigma_3 x_a f)$, $\widetilde{L}(\sigma_3 f)$ for $L = A_a, B_a, C, D_i, (\mathcal{K} \quad)_{\underline{j}}$. We denote by I_3 the set of these functions.

The *second claim* is true for $n = 0$ since by (5.36)

$$L(Y) = L_{1b}\langle \sigma_1\sigma_3\partial_b f|Y\rangle + L_{2b}\langle \sigma_1 x_b f|Y\rangle + L_3\langle \sigma_1 f|Y\rangle + \widetilde{L}(Y).$$

In this case $\varpi_{i_3} = 1$. Suppose that the *second claim* holds for $n-1$. For definiteness we will look at the case of $\mathcal{A}_a^{(n+1)}(Y)$. We have

$$\begin{aligned}\mathcal{A}_a^{(n+1)}(Y) &= \widetilde{\mathcal{A}_a}(\mathcal{K}^n Y) + (\mathcal{A}_a)_{1b}\langle \sigma_1\sigma_3\partial_b f | (\mathcal{K}^n Y)_f\rangle \\ &+ (\mathcal{A}_a)_{2b}\langle \sigma_1 x_b f | (\mathcal{K}^n Y)_f\rangle + (\mathcal{A}_a)_3\langle \sigma_1 f | (\mathcal{K}^n Y)_f\rangle .\end{aligned}$$

For definiteness let us look at the last term on the first line. Then, substituting

$$(\mathcal{K}^n Y)_f = A_a^{(n)}(Y)\partial_a f + \sigma_3[B_a^{(n)}(Y)x_a + C^{(n)}(Y)]f + D_i^{(n)}(Y)\Psi_i$$

and using the induction hypothesis, we obtain the desired result. By proceeding in the same way in all the other cases we get the *second claim*.

What is left in the proof of Lemma 5.8 is the regularity in Claim (2). The functions in (i)–(iv) are like the $G(t,z,f,\varrho)$ in the statement of Claim (2). But then Claim (2) follows by the elementary fact that, if f_0,...,f_N are smooth scalar functions, if σ varies in the set $\mathfrak{F}(\mathbb{Z}_n,\mathbb{Z}_N)$ of all the maps $\mathbb{Z}_n \to \mathbb{Z}_N$ and we consider arbitrary maps $\varepsilon_n : \mathfrak{F}(\mathbb{Z}_n,\mathbb{Z}_N) \to \{0,1\}$, then there exists a fixed $\delta > 0$ such that if $|f_j|_\infty < \delta$ for all j, then the following series converges to a smooth function:

$$\sum_{n=0}^{\infty} \sum_{\sigma\in\mathfrak{F}(\mathbb{Z}_n,\mathbb{Z}_N)} \varepsilon_n(\sigma) \prod_{j=0}^{n} f_{\sigma(j)} . \tag{5.56}$$

We sketch a proof assuming that the f_j are functions of one scalar variable. It is enough to show that the series obtained differentiating term by term in (5.56) are totally convergent. This is immediate for the 0–th derivative. The k–th derivative yields, for $|\mu| = \sum_j |\mu(j)|$, a series of the form

$$\sum_{n=0}^{\infty} A_n \text{ with } A_n = \sum_{\substack{\sigma\in\mathfrak{F}(\mathbb{Z}_n,\mathbb{Z}_N) \\ \mu\in\mathfrak{F}(\mathbb{Z}_n,\mathbb{Z}_k) \text{ s.t. } |\mu| = k}} \varepsilon_n(\sigma) \prod_{j=0}^{n} f_{\sigma(j)}^{(\mu(j))} . \tag{5.57}$$

Then we have the bound

$$\begin{aligned}|A_n| &\le (N+1)^{n+1}(n+1)^k \sup\left\{ |\prod_{j=0}^{n} f_{\sigma(j)}^{(\mu(j))}| \text{ s.t. } (\sigma,\mu) \text{ as in (5.57)} \right\} \\ &\le (N+1)^{n+1}(n+1)^k \delta^{n-k} \sup_j \|f_j\|_{W^{k,\infty}}^k .\end{aligned}$$

So $|A_n| \le \delta^{n+1-k}(N+1)^{n+1}(n+1)^k C_k^k$ for $C_k = \sup_j \|f_j\|_{W^{k,\infty}}$. Then for $(N+1)\delta_0 < 1$ and for $\delta \in (0,\delta_0)$ the series (5.57) is convergent for any k.

□

Having established the existence and a number of properties of $\mathcal{X}^t$, in Sect. 6 we prove in an abstract set up a number of results on vectorfields. After a preliminary section on the spaces Σ_ℓ, in Sect. 8 we check that it is possible to apply the theory in Sect. 6 to appropriate generalizations of $\mathcal{X}^t$.

6 Some results on abstract ODE's

We collect a number of results needed for Darboux Theorem and the method of normal forms. We will consider a system

$$\dot{x} = f(t,x)\,, \quad x(0) = \underline{x}. \tag{6.1}$$

We assume the following set up.

(1) We consider five separable Hilbert spaces $\mathbb{E}_i$ with $i = 0, 4$ s.t. $\mathbb{E}_i \subset \mathbb{E}_{i+1}$ for $i < 4$, $\mathbb{E}_i$ is dense in $\mathbb{E}_{i+1}$ and the immersion $j^{(i)}$ of $\mathbb{E}_i$ in $\mathbb{E}_{i+1}$ is compact. We denote by $(\ ,\ )_i$ resp. $\|\ \|_i$ the inner product resp. the norm in $\mathbb{E}_i$.

(2) We assume the existence of $\{j_\epsilon : \epsilon > 0\} \subset B(\mathbb{E}_{i+1}, \mathbb{E}_i)$ for $i = 0, ..., 3$ s.t.: $\|j_\epsilon \circ j^{(i)}\|_{B(\mathbb{E}_i,\mathbb{E}_i)} \le C$ for fixed C for all $\epsilon > 0$; $\lim_{\epsilon \searrow 0} j^{(i)} \circ j_\epsilon = \mathbb{1}_{\mathbb{E}_{i+1}}$ in $B(\mathbb{E}_{i+1}, \mathbb{E}_{i+1})$; $\lim_{\epsilon \searrow 0} j_\epsilon \circ j^{(i)} = \mathbb{1}_{\mathbb{E}_i}$ in $B(\mathbb{E}_i, \mathbb{E}_i)$.

(3) Let $\mathfrak{B}_i$ be the neighborhood of $0 \in \mathbb{E}_i$ defined by $\|x\|_4 < c_0$, for some fixed $c_0 > 0$. Then $f \in C^n((-3,3) \times \mathfrak{B}_i, \mathbb{E}_{i+1})$ for a $n \ge 1$.

(4) The following inequalities hold for a positive constant $C(\lambda)$ which is increasing functions of λ:

$$\|f(t,x)\|_{i+1} \le C(\|x\|_4)\|x\|_i; \tag{6.2}$$

$$|(j_\epsilon f(t,x), x)_i| \le C(\|x\|_4)\|x\|_i^2 \ \forall\, \epsilon \in (0,1) \text{ and } i; \tag{6.3}$$

$$\begin{aligned} &|(\partial_x^k j_\epsilon f(t,x)(u,v), v)_{i+1}| \le C(\|x\|_i)\|u\|_{\mathbb{E}_i^{k-1}}\|v\|_{i+1}^2 \\ &\forall\ 1 \le k \le n,\ \epsilon > 0\ ,\ i = 0, ..., 3 \text{ and } v \in \mathbb{E}_i. \end{aligned} \tag{6.4}$$

The main results of this section are the three Proposition 6.1 and 6.2.

Proposition 6.1. *$\exists$ a neighborhood $\mathcal{U}$ of 0 in $\mathfrak{B}_0 \subseteq \mathbb{E}_0$ s.t. $\forall\, \underline{x} \in \mathcal{U}$ system (6.1) has exactly one solution $x(t) \in \cap_{i=1}^2 C^{i-1}([-2,2], \mathbb{E}_i)$. $\mathcal{U}$ can be chosen to be defined by $\|\underline{x}\|_4 < \varepsilon_0$. For $\varepsilon_0 > 0$ small enough we have for a fixed C*

$$\begin{aligned} &\|x\|_{L^\infty([-2,2],\mathbb{E}_0)} \le C\|\underline{x}\|_0\,, \quad \|x\|_{W^{1,\infty}([-2,2],\mathbb{E}_1)} \le C\|\underline{x}\|_0\,, \\ &\|x\|_{L^\infty([-2,2],\mathbb{E}_4)} \le C\|\underline{x}\|_4. \end{aligned} \tag{6.5}$$

Furthermore, denoting by ϕ^t the flow associated to (6.1), we have $\phi^t(\underline{x}) \in C([-2,2], C^n(\mathcal{U}, \mathbb{E}_2))$.

We will use also a second version of the above result.

Proposition 6.2. *Assume that hypotheses (1)–(4) hold, but only with four spaces $\mathbb{E}_i$ with $i = 0, 3$ and with $\|\ \|_3$ replacing the $\|\ \|_4$ norm. Then there exists an $\varepsilon_0 > 0$ such that if $\mathcal{U}$ is the subset of $\mathbb{E}_0$ defined by $\|\underline{x}\|_3 < \varepsilon_0$, system (6.1)*

has exactly one solution $x(t) \in \cap_{i=1}^{2} C^{i-1}([-2,2],\mathbb{E}_i)$. The following inequalities hold for a fixed C

$$\begin{aligned} &\|x\|_{L^\infty([-2,2],\mathbb{E}_0)} \le C\|\underline{x}\|_0\,, \quad \|x\|_{W^{1,\infty}([-2,2],\mathbb{E}_1)} \le C\|\underline{x}\|_0\,, \\ &\|x\|_{L^\infty([-2,2],\mathbb{E}_3)} \le C\|\underline{x}\|_3. \end{aligned} \tag{6.6}$$

Furthermore, if we denote by ϕ^t the flow associated to (6.1)*, then $\partial_t^i \phi^t(\underline{x}) \in C([-2,2], C^n(\mathcal{U},\mathbb{E}_{2+i}))$ for $i=0,1$.*

Proof. The proof is the same of that of Proposition 6.1 with one minor modification. That is, in Lemma 6.3 below, set $X=\mathbb{E}_3$ instead of $X=\mathbb{E}_4$. The corresponding inequalities and their proofs are exactly the same. □

Proof of Proposition 6.1. The proof tailored on standard arguments, see [Ta] after p. 360, but in the absence of an obvious reference we review it.

We consider systems

$$\dot{x}_\epsilon = j_\epsilon f(t,x_\epsilon)\,, \quad x_\epsilon(0)=\underline{x}. \tag{6.7}$$

We have the following lemma.

Lemma 6.3. *There is $\varepsilon_0>0$ s.t., $\forall \underline{x}\in\mathbb{E}_0$ with $\|\underline{x}\|_4<\varepsilon_0$, system* (6.7) *has exactly one solution $x_\epsilon(t)\in C^1([-2,2],\mathbb{E}_0)$. In particular there is a fixed C s.t. for all $\epsilon\in(0,1)$*

$$\begin{aligned} &\|x_\epsilon\|_{L^\infty([-2,2],\mathbb{E}_0)\cap W^{1,\infty}([-2,2],\mathbb{E}_1)} \le C\|\underline{x}\|_0, \\ &\|x_\epsilon\|_{L^\infty([-2,2],\mathbb{E}_4)} \le C\|\underline{x}\|_4. \end{aligned} \tag{6.8}$$

Proof. By hypothesis have $\|j_\epsilon \partial_x^k f(t,x)\|_0 \le C(\epsilon,x)$ for all $k=0,...,n$ and $x\in\mathfrak{B}_0$. This implies that for $\underline{x}\in\mathfrak{B}_0$ the conclusions of Lemma 6.3 hold in some interval $(-a_\epsilon(\underline{x}),a_\epsilon(\underline{x}))$. We want to show that $a_\epsilon(\underline{x})>2$ if $\|\underline{x}\|_X<\varepsilon_0$ for $\varepsilon_0>0$ small enough, where $X=\mathbb{E}_4$. We consider by (6.3)

$$\left|\frac{d}{dt}\|x_\epsilon\|_0^2\right| = 2|(j_\epsilon f(t,x_\epsilon),x_\epsilon)_0| \le 2C(\|x_\epsilon\|_X)\|x_\epsilon\|_0^2.$$

By Gronwall we get $\|x_\epsilon\|_0 \le e^{|t|C(1)}\|\underline{x}\|_0$ as long as $\|x_\epsilon\|_X\le 1$ in $[-|t|,|t|]$. We claim that the latter holds for $|t|\le \frac{1-\log\varepsilon_0}{C(1)}$. By (6.3) we have

$$\frac{d}{dt}\|x_\epsilon\|_X^2 = 2(j_\epsilon f(t,x_\epsilon),x_\epsilon)_X \le 2C(\|x_\epsilon\|_X)\|x_\epsilon\|_X^2.$$

For ε_0 small enough, using Gronwall we obtain (6.8) and $|t|\ge 2$. By equation (6.7) and by (6.2) we have $\|\dot{x}_\epsilon\|_1 \le C'\|\underline{x}\|_0$ for some fixed C'. So Lemma 6.3 is proved. □

We now exploit Lemma 6.3 to prove the first part of Proposition 6.1 estimates (6.5). The argument is routine. Given a sequence $\epsilon_\nu \searrow 0$ then $\{x_{\epsilon_\nu}\}$ admits a subsequence convergent in $C([-2,2],\mathbb{E}_1)$ by Ascoli Arzéla. We can assume it is the whole sequence. We denote by $x(t)$ the limit. Since we have necessarily $x_{\epsilon_\nu}\to x$ in $C([-2,2],\mathbb{E}_j)$ for $j>1$, (6.8) yields the third inequality

in (6.5). Notice that $x(t)$ is the weak limit of $x_{\epsilon_\nu}(t)$ in $\mathbb{E}_0$ for any $|t| \le 2$. By Fathou this yields the first inequality in (6.5). We have $f(t, x_{\epsilon_\nu}) \to f(t,x)$ in $C([-2,2], \mathbb{E}_2)$. We claim we have $j_{\epsilon_\nu} f(t, x_{\epsilon_\nu}) \to f(t,x)$ in $C([-2,2], \mathbb{E}_2)$. Indeed, by $\lim_{\epsilon \searrow 0} j_\epsilon = j$ in $B(\mathbb{E}_1, \mathbb{E}_2)$ and by $\|f(t, x_{\epsilon_\nu})\|_{\mathbb{E}_1} \le C\|\underline{x}\|_{\mathbb{E}_0}$, it follows that in $\mathbb{E}_2$ and uniformly in $t \in [-2,2]$, we have

$$j_{\epsilon_\nu} f(t, x_{\epsilon_\nu}) = (j_{\epsilon_\nu} - j f(t, x_{\epsilon_\nu}) + j^{(1)} f(t, x_{\epsilon_\nu}) \to f(t,x).$$

This implies $\dot{x} \in C([-2,2], \mathbb{E}_2)$ where is the limit of $\dot{x}_{\epsilon_\nu}$ and satisfies the equation (6.1). By Fathou as before this yields the second inequality in (6.5). Suppose $y \in \cap_{i=1}^2 C^{i-1}([-2,2], \mathbb{E}_i)$ is a solution of (6.1). Then by (6.4) for $k=1$ we have for $\delta x = y - x$

$$\frac{d\|\delta x\|_2^2}{dt} = 2\int_0^1 (\partial_x f(t, (1-\tau)x + \tau y)\delta x, \delta x)_2 \le C(\|x\|_1, \|y\|_1)\|\delta x\|_2^2. \tag{6.9}$$

Notice indeed that, by $\lim_{\epsilon \searrow 0} j_\epsilon = j^{(i)}$ in $B(\mathbb{E}_i, \mathbb{E}_{i+1})$, (6.4) implies

$$|(\partial_x^k f(t,x)(u,v), v)_{i+1}| \le C(\|x\|_i)\|u\|_{\mathbb{E}_i^{k-1}} \|v\|_{i+1}^2. \tag{6.10}$$

By Gronwall, (6.9) implies $x \equiv y$. This concludes the proof of the first part of Prop. 6.1 and of (6.5).

We now turn to the proof of the last sentence of Prop. 6.1. $\mathcal{U}$ will be the neighborhood of 0 in $\mathbb{E}_0$ defined by $\|\underline{x}\|_4 < \varepsilon_0$. We have proved that $\phi^t(\underline{x}) = \lim_{\epsilon \searrow 0} \phi_\epsilon^t(\underline{x})$ in $C([-2,2], \mathbb{E}_1)$, with ϕ_ϵ^t the flow associated to (6.7). We have $\phi_\epsilon^t(\underline{x}) \in C^n([-2,2] \times \mathcal{U}, \mathbb{E}_0)$.

Lemma 6.4. *$\phi^t \in C(\mathcal{U}, \mathbb{E}_1)$ for all $t \in [-2,2]$.*

Proof. Given $\underline{x}, \underline{y} \in \mathcal{U}$ set $\delta\phi_\epsilon = \phi_\epsilon^t(\underline{x}) - \phi_\epsilon^t(\underline{y})$. By (6.4) and the first part of Proposition 6.1

$$\frac{d}{dt}\|\delta\phi_\epsilon\|_1^2 = 2(j_\epsilon(f(t, \phi_\epsilon^t(\underline{x})) - f(t, \phi_\epsilon^t(\underline{y}))), \delta\phi_\epsilon)_1 \le C\|\delta\phi_\epsilon\|_1^2.$$

This implies $\|\phi_\epsilon^t(\underline{x}) - \phi_\epsilon^t(\underline{y})\|_1 \le C'\|\underline{x} - \underline{y}\|_1$ for a fixed C'. For $\epsilon \searrow 0$ this yields for fixed C''

$$\|\phi^t(\underline{x}) - \phi^t(\underline{y})\|_1 \le C'\|\underline{x} - \underline{y}\|_1 \le C''\|\underline{x} - \underline{y}\|_0. \tag{6.11}$$

$\square$

Lemma 6.5. *We have $\|\partial_{\underline{y}}^l \phi_\epsilon^t(\underline{y})\|_{B^l(\mathbb{E}_0, \mathbb{E}_1)} + \|\partial_t \partial_{\underline{y}}^l \phi_\epsilon^t(\underline{y})\|_{B^l(\mathbb{E}_0, \mathbb{E}_2)} \le C$ for a fixed C and for all $1 \le l \le n$, $\underline{y} \in \mathcal{U}$ and $\epsilon \in (0,1)$.*

Proof. We have

$$\partial_t \partial_{\underline{y}} \phi_\epsilon^t(\underline{y}) = j_\epsilon \partial_\phi f(t, \phi_\epsilon^t(\underline{y}))\partial_{\underline{y}} \phi_\epsilon^t(\underline{y}), \quad \partial_{\underline{y}} \phi_\epsilon^0(\underline{y}) = 1\!\!1. \tag{6.12}$$

For $l > 1$ we have $\partial^l_{\underline{y}} \phi^0_\epsilon(\underline{y}) = 0$ and, succinctly,

$$\begin{aligned}
&\partial_t \partial^l_{\underline{y}} \phi^t_\epsilon(\underline{y}) = j_\epsilon \partial_\phi f(t, \phi^t_\epsilon(\underline{y})) \partial^l_{\underline{y}} \phi^t_\epsilon(\underline{y}) + \\
&\mathrm{Sym} \sum_{k=2}^{l} \sum_{|\alpha|=l} j_\epsilon \partial^k_\phi f(t, \phi^t_\epsilon(\underline{y})) \frac{l!}{\alpha!} \partial^{\alpha_1}_{\underline{y}} \phi^t_\epsilon(\underline{y}) ... \partial^{\alpha_k}_{\underline{y}} \phi^t_\epsilon(\underline{y}),
\end{aligned} \tag{6.13}$$

with $|\alpha| = \sum_j \alpha_j$ and $\alpha! = \prod_j \alpha_j!$ and Sym an appropriate symmetrization, see [N] p.7. Fix $v \in \mathbb{E}_0$. By (6.4) and for $\partial_{\underline{y}} \phi^t_\epsilon = \partial_{\underline{y}} \phi^t_\epsilon(\underline{y})$,

$$\frac{d}{dt} \|\partial_{\underline{y}} \phi^t_\epsilon v\|^2_1 = 2(j_\epsilon \partial_\phi f(t, \phi^t_\epsilon) \partial_{\underline{y}} \phi^t_\epsilon v, \partial_{\underline{y}} \phi^t_\epsilon v)_1 \leq C \|\partial_{\underline{y}} \phi^t_\epsilon v\|^2_1. \tag{6.14}$$

Since C is independent of v, we obtain $\|\partial_{\underline{y}} \phi^t_\epsilon(\underline{y})\|_{B(\mathbb{E}_0, \mathbb{E}_1)} \leq C_1$ for some fixed C_1 by Gronwall. We assume $\|\partial^k_{\underline{y}} \phi^t_\epsilon(\underline{y})\|_{B^k(\mathbb{E}_0, \mathbb{E}_1)} \leq C$ for $k < l$ by induction. Let $K(t, \epsilon, \underline{y})$ be the second line of (6.13). Then $\|K(t, \epsilon, \underline{y})\|_{B^l(\mathbb{E}_0, \mathbb{E}_1)} \leq C$ by induction. Fix $v \in \mathbb{E}^l_0$. Then, proceeding as for $l = 1$ we get

$$\frac{d}{dt} \|\partial^l_{\underline{y}} \phi^t_\epsilon(\underline{y}) v\|^2_1 \leq C \|\partial^l_{\underline{y}} \phi^t_\epsilon(\underline{y}) v\|_1 + C \|K(t, \epsilon, \underline{y}) v\|_1. \tag{6.15}$$

By Gronwall we get $\|\partial^l_{\underline{y}} \phi^t_\epsilon(\underline{y}) v\|_1 \leq C_l \|v\|_{\mathbb{E}^l_0}$ and so $\|\partial^l_{\underline{y}} \phi^t_\epsilon(\underline{y}) v\|_{B^l(\mathbb{E}_0, \mathbb{E}_1)} \leq C_l$ since the constants C in (6.15) and C_l do not depend on v. By equations (6.12)–(6.13) we obtain $\|\partial_t \partial^l_{\underline{y}} \phi^t_\epsilon(\underline{y}) v\|_{B^l(\mathbb{E}_0, \mathbb{E}_2)} \leq C'_l$. $\square$

The natural embedding $B^l(\mathbb{E}_i, \mathbb{E}_{i+1}) \hookrightarrow B^l(\mathbb{E}_i, \mathbb{E}_{i+2})$ is compact. This implies that for any fixed y and any sequence $\epsilon_\nu \searrow 0$ there is a subsequence $\partial^l_{\underline{y}} \phi^t_{\epsilon_\nu}(\underline{y})$ convergent in $C([-2, 2], B^l(\mathbb{E}_0, \mathbb{E}_2))$ to a $g^{(l)}(t, y)$. $\partial_t g^{(1)}(t, \underline{y})$ exists by (6.12), with $\partial_t \partial_{\underline{y}} \phi^t_{\epsilon_\nu}(\underline{y})$ convergent to it in $C([-2, 2], B(\mathbb{E}_0, \mathbb{E}_3))$ and with

$$\partial_t g^{(1)}(t, \underline{y}) = \partial_\phi f(t, \phi^t(\underline{y})) g^{(1)}(t, \underline{y}), \quad g^{(1)}(0, \underline{y}) = \mathbb{1}. \tag{6.16}$$

Suppose that $g^{(1)}_1(t, \underline{y})$ and $g^{(1)}_2(t, \underline{y})$ are two such solutions of (6.16). Fix $v \in \mathbb{E}_0$ and set $\delta g(t, \underline{y}) = g^{(1)}_1(t, \underline{y}) - g^{(1)}_2(t, \underline{y})$. Then

$$\frac{d}{dt} \|\delta g(t, \underline{y}) v\|^2_3 = 2(\partial_\phi f(t, \phi^t(\underline{y})) \delta g(t, \underline{y}) v, \delta g(t, \underline{y}) v)_3 \leq C \|\delta g(t, \underline{y}) v\|^2_3,$$

by (6.4) and the first part of Proposition 6.1. This implies $\delta g(t, \underline{y}) v = 0$ and so $g^{(1)}_1(t, \underline{y}) = g^{(1)}_2(t, \underline{y})$. For $l > 1$, by induction and a similar argument, we get a function $g^{(l)}(t, \underline{y}) \in C([-2, 2], B^l(\mathbb{E}_0, \mathbb{E}_2))$ satisfying $g^{(l)}(0, \underline{y}) = 0$ and such that $\partial_t g^{(l)}(t, \underline{y}) \in C([-2, 2], B^l(\mathbb{E}_0, \mathbb{E}_3))$ satisfies

$$\begin{aligned}
&\partial_t g^{(l)}(t, \underline{y}) = \partial_\phi f(t, \phi^t(\underline{y})) g^{(l)}(t, \underline{y}) + \\
&\mathrm{Sym} \sum_{k=2}^{l} \sum_{|\alpha|=l} j_\epsilon \partial^k_\phi f(t, \phi^t_\epsilon(\underline{y})) \frac{l!}{\alpha!} g^{(\alpha_1)}(t, \underline{y}) ... g^{(\alpha_k)}(t, \underline{y}),
\end{aligned} \tag{6.17}$$

where in the second line we have only $g^{(\ell)}(t,\underline{y})$ with $\ell < l$, which can be assumed uniquely defined by induction. By repeating the previous argument we get uniqueness also for $k = l$.

Lemma 6.6. *For any t the map $\phi^t : \mathcal{U} \to \mathbb{E}_2$ is Frechét differentiable with $\partial_{\underline{y}}\phi^t(\underline{y}) = g^{(1)}(t,\underline{y})$. We have $\partial_t^i \partial_{\underline{y}}\phi^t \in C([-2,2]\times\mathcal{U}, B(\mathbb{E}_0,\mathbb{E}_{i+2}))$ for $i = 0,1$.*

Proof. For fixed $\underline{x}, \underline{y} \in \mathcal{U}$ set $\delta\underline{x} = \underline{y} - \underline{x}$ and $\delta\phi = \phi^t(\underline{y}) - \phi^t(\underline{x})$. Then

$$\partial_t[\delta\phi - g^{(1)}(t,\underline{x})\delta\underline{x}] = f(t,\phi^t(\underline{y})) - f(t,\phi^t(\underline{x})) - \partial_\phi f(t,\phi^t(\underline{x}))g^{(1)}(t,\underline{x})\delta\underline{x}.$$

Then, for $g^{(1)} = g^{(1)}(t,\underline{x})$, we have

$$\begin{aligned}&\frac{d}{dt}\|\delta\phi - g^{(1)}\delta\underline{x}\|_2^2 = 2(\partial_\phi f(t,\phi^t(\underline{x}))(\delta\phi - g^{(1)}\delta\underline{x}), \delta\phi - g^{(1)}\delta\underline{x})_2\\ &+\int_0^1 (\partial_\phi^2 f(t,(1-\tau)\phi^t(\underline{x}) + \tau\phi^t(\underline{y}))(\delta\phi)^2, \delta\phi - g^{(1)}\delta\underline{x})_2\, d\tau.\end{aligned}$$

The last line is less than $C\|\delta\phi\|_1^2\|\delta\phi - g^{(1)}\delta\underline{x}\|_2$. Hence by (6.11)

$$\begin{aligned}&\frac{d}{dt}\|\delta\phi - g^{(1)}\delta\underline{x}\|_2 \le C\|\delta\phi - g^{(1)}\delta\underline{x}\|_2 + C\|\delta\phi\|_1^2\\ &\le C\|\delta\phi - g^{(1)}\delta\underline{x}\|_2 + C_1\|\delta\underline{x}\|_1^2.\end{aligned}$$

This yields $\|\delta\phi - g^{(1)}\delta\underline{x}\|_2 = o(\|\delta\underline{x}\|_0)$. Hence $\phi^t : \mathcal{U} \to \mathbb{E}_2$ is Frechét differentiable with $\partial_y\phi^t(\underline{y}) = g^{(1)}(t,\underline{y})$. Set $\delta g^{(1)} = g^{(1)}(t,\underline{y}) - g^{(1)}(t,\underline{x})$. Set $\delta\underline{x} = \underline{y} - \underline{x}$ like above. Then

$$\partial_t\delta g^{(1)} = \partial_\phi f(t,\phi^t(\underline{y}))g^{(1)}(t,\underline{y}) - \partial_\phi f(t,\phi^t(\underline{x}))g^{(1)}(t,\underline{x}).$$

Then for fixed $v \in \mathbb{E}_0$

$$\begin{aligned}&\frac{d}{dt}\|\delta g^{(1)}v\|_2^2 = 2(\partial_\phi f(t,\phi^t(\underline{x}))\delta g^{(1)}v, \delta g^{(1)}v)_2\\ &+2((\partial_\phi f(t,\phi^t(\underline{y})) - \partial_\phi f(t,\phi^t(x))g^{(1)}(t,\underline{y})v, \delta g^{(1)}v)_2.\end{aligned}$$

By (6.10), for a fixed C we have

$$\frac{d}{dt}\|\delta g^{(1)}v\|_2 \le C\|\delta g^{(1)}v\|_2 + C\|\delta\underline{x}\|_0\|v\|_0.$$

Hence $\|g^{(1)}(t,\underline{y}) - g^{(1)}(t,\underline{x})\|_{B(\mathbb{E}_0,\mathbb{E}_2)} \le C'\|\underline{y} - \underline{x}\|_0$ for a fixed C'. This and $g^{(1)}(t,\underline{x}) \in C([-2,2], B(\mathbb{E}_0,\mathbb{E}_2))$ for all $\underline{x}$ imply $g^{(1)} \in C([-2,2]\times\mathcal{U}, B(\mathbb{E}_0,\mathbb{E}_2))$. We obtain $\partial_t g^{(1)} \in C([-2,2]\times\mathcal{U}, B(\mathbb{E}_0,\mathbb{E}_3))$ by (6.16). $\square$

Lemma 6.7. *$\partial_{\underline{y}}^{l-1}\phi^t : \mathcal{U} \to \mathbb{E}_2$ is Frechét differentiable for any t and $1 \le l \le n$ with $\partial_{\underline{y}}^l\phi^t(\underline{y}) = g^{(l)}(t,\underline{y})$. We have $\partial_t^i\partial_{\underline{y}}^l\phi^t \in C([-2,2]\times\mathcal{U}, B^l(\mathbb{E}_0,\mathbb{E}_{i+2}))$ for $i = 0,1$.*

Proof. Case $l = 1$ is proved in Lemma 6.6. Consider $l > 1$. By induction we can assume: $\phi^t(\underline{x})$ admits Frechét derivatives of order $k < l$; $\partial^k_{\underline{x}}\phi^t(\underline{x}) = g^{(k)}(t, \underline{x})$; $\partial^i_t g^{(k)} \in C([-2, 2]\times\mathcal{U}, B^k(\mathbb{E}_0, \mathbb{E}_{i+2}))$ for $i = 0, 1$. Fix now $\underline{x}, \underline{y} \in \mathcal{U}$ and $v \in \mathbb{E}_0^{l-1}$. Set $\delta\underline{x} = \underline{y} - \underline{x}$, $g^{(l)} = g^{(l)}(t, \underline{x})$ and $\delta g^{(l-1)} = g^{(l-1)}(t, \underline{y}) - g^{(l-1)}(t, \underline{x})$. Then

$$\begin{aligned}
&\partial_t[\delta g^{(l-1)} v - g^{(l)}(v, \delta\underline{x})] = \partial_\phi f(t, \phi^t(\underline{x}))[\delta g^{(l-1)} v - g^{(l)}(v, \delta\underline{x})] \\
&+ (\partial_\phi f(t, \phi^t(\underline{y})) - \partial_\phi f(t, \phi^t(\underline{x})))\partial^{l-1}_y \phi^t(\underline{y}) \cdot v + F(z)\big]^{z=\underline{y}}_{z=\underline{x}} \cdot v \\
&- \mathrm{Sym} \sum_{k=2}^{l} \sum_{|\alpha|=l} \partial^k_\phi f(t, \phi^t(\underline{x})) \frac{l!}{\alpha!} \partial^{\alpha_1}_{\underline{x}} \phi^t(\underline{x}) ... \partial^{\alpha_k}_{\underline{x}} \phi^t(\underline{x}) \cdot (v, \delta\underline{x}),
\end{aligned} \tag{6.18}$$

where $|\alpha| = \sum_{j=1}^k \alpha_j$, $\alpha! = \prod_{j=1}^k \alpha_j!$ and where

$$F(z) := \mathrm{Sym} \sum_{k=2}^{l-1} \sum_{|\alpha|=l-1} \partial^k_\phi f(t, \phi^t(z)) \frac{(l-1)!}{\alpha!} \partial^{\alpha_1}_z \phi^t(z) ... \partial^{\alpha_k}_z \phi^t(z). \tag{6.19}$$

Notice that $|\alpha_j| \leq l - 1$ (resp. $|\alpha_j| \leq l - 2$) for all j in (6.18) (resp. (6.19)). The last two lines in (6.18) can be written as

$$\begin{aligned}
&F(z)\big]^{z=\underline{y}}_{z=\underline{x}} \cdot v + o(\delta\underline{x}) + \partial^2_\phi f(t, \phi^t(\underline{x}))(\partial_{\underline{x}} \phi^t(\underline{x})\delta\underline{x})\, \partial^{l-1}_{\underline{x}} \phi^t(\underline{x}) \cdot v \\
&- \mathrm{Sym} \sum_{k=2}^{l} \sum_{|\alpha|=l} \partial^k_\phi f(t, \phi^t(\underline{x})) \frac{l!}{\alpha!} \partial^{\alpha_1}_{\underline{x}} \phi^t(\underline{x}) ... \partial^{\alpha_k}_{\underline{x}} \phi^t(\underline{x}) \cdot (v, \delta\underline{x}),
\end{aligned} \tag{6.20}$$

where $\|o(\delta\underline{x})\|_2 = \|v\|_{\mathbb{E}_0^{l-1}} o(\|\delta\underline{x}\|_0)$. The last term in the first line cancels with a corresponding one in the second. What remain in the last line (6.20) is $-(F'(\underline{x})\delta\underline{x}) \cdot v$, where we know by induction that $F(z)$ is Frechét differentiable. Hence all (6.20) is $o(\delta\underline{x})$. Then by (6.4) and Gronwall we get

$$\|\delta g^{(l-1)} - g^{(l)}(t, \underline{x})\delta\underline{x}\|_{B^{l-1}(\mathbb{E}_0,\mathbb{E}_2)} = o(\|\delta\underline{x}\|_0).$$

So $g^{(l-1)}(t, \cdot) : \mathcal{U} \to \mathbb{E}_2$ is Frechét differentiable and $\partial_x g^{(l-1)}(t, \underline{x}) = g^{(l)}(t, \underline{x})$. By the usual argument one shows that $\|g^{(l)}(t, \underline{y}) - g^{(l)}(t, \underline{x})\|_{B^l(\mathbb{E}_0,\mathbb{E}_2)} \leq C'\|\underline{y} - \underline{x}\|_0$ for a fixed C'. This and $g^{(l)}(t, \underline{x}) \in C([-2, 2], B^l(\mathbb{E}_0, \mathbb{E}_2))$ for fixed $\underline{x}$ imply that $g^{(l)} \in C([-2, 2] \times \mathcal{U}, B^l(\mathbb{E}_0, \mathbb{E}_2))$. We obtain $\partial_t g^{(l)} \in C([-2, 2] \times \mathcal{U}, B^l(\mathbb{E}_0, \mathbb{E}_3))$ by (6.17).

□

The last statement of Proposition 6.1 is proved.

□

7 Some facts on the spaces Σ_l

We need some preliminary information on the space Σ_ℓ defined by the norm (1.8). Consider the space Σ'_ℓ with norm

$$\|U\|^2_{\Sigma'_\ell} := \|U\|^2_{H^n} + \sum_{|\alpha|\le \ell} \| \, x^\alpha U\|^2_{L^2} < \infty. \tag{7.1}$$

Lemma 7.1. *We have $\Sigma_\ell = \Sigma'_\ell$. Σ_ℓ is preserved by the flow of* (2.6).

Proof. The second statement follows from the first by the fact that Σ'_ℓ is preserved by the flow of (2.6), see [MS].

The fact that $\Sigma'_\ell \subseteq \Sigma_n$ follows from Proposition 2 [MS]. We prove $\Sigma'_\ell \supseteq \Sigma_\ell$ by induction. Set $T_a := \mathrm{i}\partial_a + \mathrm{i}x_a$. We have $T_a^* T_a = -\partial_a^2 + x_a^2 - 1$. Then

$$\|U\|^2_{\Sigma_1} \approx \|U\|^2_{L^2} + \langle(-\Delta + |x|^2)U|\overline{U}\rangle \approx \|U\|^2_{\Sigma'_1} \tag{7.2}$$

by the fact that $\langle(-\Delta+1)U|\overline{U}\rangle$ defines H^1. So $\Sigma_1 = \Sigma'_1$. Suppose by induction $\Sigma_{\ell-1} = \Sigma'_{\ell-1}$. Let $|\alpha| = \ell - 1$. Then

$$\begin{aligned} &\|\partial_x^\alpha U\|^2_{H^1} \lesssim \|\partial_x^\alpha U\|^2_{L^2} + \langle(-\Delta+|x|^2)\partial_x^\alpha U|\partial_x^\alpha \overline{U}\rangle \approx \|\partial_x^\alpha U\|^2_{L^2} \\ &+ \sum_{a=1}^3 \|(\mathrm{i}\partial_a + \mathrm{i}x_a)\partial_x^\alpha U\|^2_{L^2} \le \|\partial_x^\alpha U\|^2_{L^2} \\ &+ \sum_{a=1}^3 \|\partial_x^\alpha(\mathrm{i}\partial_a + \mathrm{i}x_a)U\|^2_{L^2} + \sum_{a=1}^3 \|[\mathrm{i}\partial_a + \mathrm{i}x_a, \partial_x^\alpha]U\|^2_{L^2} \lesssim \|U\|^2_{\Sigma_\ell}, \end{aligned} \tag{7.3}$$

by induction and by $\sum_{a=1}^3 \|[\mathrm{i}\partial_a + \mathrm{i}x_a, \partial_x^\alpha]U\|_{L^2} \lesssim \|u\|_{H^{\ell-1}}$. For the same α, using the Fourier transform we see that (7.3) implies also

$$\sum_{a=1}^3 \|x_a x^\alpha U\|^2_{L^2} \lesssim \|U\|^2_{\Sigma_\ell}. \tag{7.4}$$

Then (7.3)–(7.4) imply $\|U\|_{\Sigma'_\ell} \lesssim \|U\|_{\Sigma_\ell}$ and so $\Sigma_\ell \subseteq \Sigma'_\ell$. Since we know already $\Sigma_\ell \supseteq \Sigma'_\ell$, they are equal. ☐

Definition 7.2. For any positive integer ℓ we denote by $\Sigma_{-\ell}$ the space formed by the V such that the map $U \to \langle U|\overline{V}\rangle$ is in $B(\Sigma_\ell, \mathbb{C})$. We we also set $\Sigma_0 := L^2$.

Definition 7.2 yields a natural Banach structure on $\Sigma_{-\ell}$. Notice that we have found two distinct but equivalent norms in Σ_ℓ. A third one comes from Claim (4) in Lemma 7.3. In the subsequent proofs we will pick in the proofs from time to time the norms which are most convenient, the statements being unsensible to the particular choice. There will also be a corresponding implicit choice of inner products.

Lemma 7.3. *Set $h = -\Delta + |x|^2$. The following facts hold.*

(1) The map $(h+1)^{\ell}:\Sigma_\ell\to\Sigma_{-\ell}$ is an isomorphism between Σ_ℓ and $\Sigma_{-\ell}$ for any $\ell\in\mathbb{N}$.

(2) $\|U\|_{\Sigma_\ell}\approx\|(h+1)^{\frac{\ell}{2}}U\|_{L^2}$ for any $\ell\in\mathbb{Z}$.

(3) The map $(h+1)^{\frac{j}{2}}:\Sigma_\ell\to\Sigma_{\ell-j}$ is an isomorphism for any $(\ell,j)\in\mathbb{Z}^2$.

(4) The map $(h+1)^{\frac{j}{2}}:\Sigma_\ell\to\Sigma_{\ell-j}$ is an isomorphism for any $(\ell,j)\in\mathbb{R}^2$ if we define Σ_ℓ for $\ell\in\mathbb{R}$ setting $\|U\|_{\Sigma_\ell}:=\|(h+1)^{\frac{\ell}{2}}U\|_{L^2}$.

Proof. The proof of (1) is an easy consequence of $T_a^*T_a=-\partial_a^2+x_a^2-1$ and $[T_a,T_b^*]=2\delta_{ab}$ and is skipped. Proofs of the other claims are elementary. □

We will consider the following mollifier:

$$J_\varepsilon=(1-\varepsilon\Delta+\varepsilon|x|^2)^{-1}\text{ for }\varepsilon>0. \tag{7.5}$$

Lemma 7.4. *Let $s>0$. the following facts hold.*

(1) Denote by $j:\Sigma_\ell\to\Sigma_{\ell-s}$ the natural embedding. Then $\lim_{\epsilon\searrow 0}J_\epsilon^{\frac{s}{2}}=j$ in $B(\Sigma_\ell,\Sigma_{\ell-s})$.

(2) We have $J_\epsilon^{\frac{s}{2}}:\Sigma_\ell\to\Sigma_{\ell+s}$ with $\|J_\epsilon^{\frac{s}{2}}\|_{B(\Sigma_\ell,\Sigma_{\ell+s})}\le C\epsilon^{-\frac{s}{2}}$.

Proof. Set $h=-\Delta+|x|^2$. For any $U\in\Sigma_\ell$ there is a fixed C s.t.

$$\begin{aligned}&\|(1-J_\epsilon^{\frac{s}{2}})U\|_{\Sigma_{\ell-s}}=\|(h+1)^{\frac{\ell-s}{2}}(1-J_\epsilon^{\frac{s}{2}})U\|_{L^2}\\&=\|(h+1)^{\frac{\ell-s}{2}}\left(1-\left(1-\frac{\epsilon h}{\epsilon h+1}\right)^{\frac{s}{2}}\right)U\|_{L^2}\le C\epsilon\|(h+1)^{\frac{\ell}{2}}U\|_{L^2}.\end{aligned}$$

So $J_\epsilon^{\frac{s}{2}}-j=O(\epsilon)$ in $B(\Sigma_\ell,\Sigma_{\ell-s})$. The second claim follows by the Spectral Theorem and $(1+r)^{\frac{\ell+s}{2}}(1+\epsilon r)^{-\frac{s}{2}}\le\epsilon^{-\frac{s}{2}}(1+r)^{\frac{\ell}{2}}$ for any $r\ge0$. □

Lemma 7.5. *For any $\ell\in\mathbb{Z}$ we have $x_a,\partial_a\in B(\Sigma_\ell,\Sigma_{\ell-1})$*

Proof. For $\ell>0$, $f\in\Sigma_\ell$ and $|\alpha|\le\ell-1$ we have

$$\|(\mathrm{i}\partial+\mathrm{i}x)^\alpha\partial_af\|_{L^2}\le\|\partial_a(\mathrm{i}\partial+\mathrm{i}x)^\alpha f\|_{L^2}+\|[(\mathrm{i}\partial+\mathrm{i}x)^\alpha,\partial_a]f\|_{L^2}\le C\|f\|_{\Sigma_\ell},$$

where we are using (7.3) and the fact that $[(\mathrm{i}\partial+\mathrm{i}x)^\alpha,\partial_a]$ is a linear combination of $(\mathrm{i}\partial+\mathrm{i}x)^\beta$ with $|\beta|=|\alpha|-1$. The case with x_a is seen to be equivalent, through the Fourier transform. The case with $\ell\le0$ follows by duality. □

Lemma 7.6. *There is a fixed $C>0$ s.t. $\forall\,\varepsilon\in(0,1)$ for $T=x_a,\partial_a$ and $\forall\,\ell\in\mathbb{Z}$ we have $\varepsilon(\|TJ_\varepsilon\|_{B(\Sigma_\ell,\Sigma_\ell)}+\|J_\varepsilon T\|_{B(\Sigma_\ell,\Sigma_\ell)})<C$.*

Proof. By Lemma 7.4 and Lemma 7.5, for $T=x_a,\partial_a$ we have $\epsilon\|J_\epsilon Tf\|_{\Sigma_\ell}\le C\|Tf\|_{\Sigma_{\ell-1}}\le C'\|f\|_{\Sigma_\ell}$. Similarly $\epsilon\|TJ_\epsilon f\|_{\Sigma_\ell}\le C\epsilon\|J_\epsilon f\|_{\Sigma_{\ell+1}}\le C'\|f\|_{\Sigma_\ell}$. □

Lemma 7.7. *For any $\ell \in \mathbb{Z}$ and $n \in \mathbb{N}$ there is a fixed $C > 0$ s.t. $\forall\ \varepsilon \in (0,1)$ we have $\|[T, J_\varepsilon^n]\|_{B(\Sigma_{2\ell},\Sigma_{2\ell})} < C$ with $T = x_a, \partial_a$.*

Proof. We have $[\partial_a, J_\varepsilon^n] = -(\epsilon h+1)^{-n}[\partial_a, (\epsilon h+1)^{-n}](\epsilon h+1)^{-n}$. It is elementary that this is a sum of terms $-2\epsilon(\epsilon h+1)^{j-n}x_a(\epsilon h+1)^{k-n}$ with $j+k = n-1$. Then for $\ell \geq 0$

$$\begin{aligned}&\|\epsilon(\epsilon h+1)^{j-n}x_a(\epsilon h+1)^{k-n}f\|_{\Sigma_\ell} \leq C\|\epsilon x_a(\epsilon h+1)^{k-n}f\|_{\Sigma_\ell}\\ &\leq C_1\|\epsilon(\epsilon h+1)^{k-n}f\|_{\Sigma_{\ell+1}} \leq C_2\|(\epsilon h+1)^{k-n+1}f\|_{\Sigma_\ell} \leq C_3\|f\|_{\Sigma_\ell}.\end{aligned}$$

The other estimates can be proved similarly. The case $\ell < 0$ follows by duality. □

Lemma 7.8. *For any $n \in \mathbb{N}$ there is a fixed $C > 0$ s.t. $\forall\ \varepsilon \in (0,1)$ we have $\|[T, J_\varepsilon^n]\|_{B(H^1,H^1)} < C$ with $T = x_a, \partial_a$.*

Proof. We know that $\|[T, J_\varepsilon^n]\|_{B(L^2,L^2)} < C$ from Lemma 7.7 for $\ell = 0$. Proceeding as above we need to show that terms like the following ones are in $B(H^1, L^2)$:

$$\begin{aligned}&\epsilon\partial_b(\epsilon h+1)^{j-n}x_a(\epsilon h+1)^{k-n} = \epsilon[\partial_b, (\epsilon h+1)^{j-n}]x_a(\epsilon h+1)^{k-n}\\ &+\delta_{ab}\epsilon(\epsilon h+1)^{-n-1} + \epsilon(\epsilon h+1)^{j-n}x_a[\partial_b, (\epsilon h+1)^{k-n}]\\ &+\epsilon(\epsilon h+1)^{j-n}x_a(\epsilon h+1)^{k-n}\partial_b.\end{aligned}$$

All the terms in the rhs except for the last one are in $B(L^2, L^2)$ with norm bounded uniformly in ϵ.For the last one the same holds in $B(H^1, L^2)$. □

Lemma 7.9. *Let $\ell \in \mathbb{Z}$ and let $n \in \mathbb{N}$ such that $n+\ell \geq 1$. Then there are fixed constants C_ℓ and C such that for $T = \partial_a, x_a$*

$$|(f, J_\epsilon^{2n}Tf)_{\Sigma_{2\ell}}| \leq C\|f\|^2_{\Sigma_{2\ell}}, \quad |(f, J_\epsilon^{2n}Tf)_{H^1}| \leq \|f\|^2_{H^1}. \tag{7.6}$$

Proof. Set $h = -\Delta + |x|^2$ and $X = \Sigma_{2\ell}, H^1$.

$$\begin{aligned}&((\epsilon h+1)^{-2n}\partial_a f, f)_X = ([(\epsilon h+1)^{-n}, \partial_a]f, (\epsilon h+1)^{-n}f)_X\\ &+(\partial_a(\epsilon h+1)^{-n}f, (\epsilon h+1)^{-n}f)_X.\end{aligned} \tag{7.7}$$

We have

$$|([(\epsilon h+1)^{-n}, \partial_a]f, (\epsilon h+1)^{-n}f)_X| \leq \|f\|_X\|[(\epsilon h+1)^{-n}, \partial_a]f\|_X \leq C\|f\|^2_X \tag{7.8}$$

by Lemmas 7.7 and 7.8. We have for $\ell \in \mathbb{Z}$

$$\begin{aligned}&(\partial_a(\epsilon h+1)^{-n}f, (\epsilon h+1)^{-n}f)_{\Sigma_{2\ell}}\\ &= ((h+1)^\ell\partial_a(\epsilon h+1)^{-n}f, (h+1)^\ell(\epsilon h+1)^{-n}f)_{L^2}\\ &= ([(h+1)^\ell, \partial_a](\epsilon h+1)^{-n}f, (h+1)^\ell(\epsilon h+1)^{-n}f)_{L^2},\end{aligned}$$

where we exploited $(\partial_a g, g)_{L^2} = (\sigma_3 x_a g, g)_{L^2} = 0$ for $\overline{g} = \sigma_1 g$. The rhs is in absolute value less than $\|f\|^2_{\Sigma_\ell}$. The proof for the case $X = H^1$ is simpler. □

8 Quasilinear systems

We will apply the theory developed in Sect. 6 in two distinct forms to quasilinear systems

$$\begin{aligned} &\dot f = \mathcal{L} f + \mathcal{D} \quad , \quad \dot z = Z \\ &\mathcal{L} := \mathcal{A}_a \partial_a + (\mathcal{B}_a x_a + \mathcal{C})\sigma_3. \end{aligned} \tag{8.1}$$

with $\mathcal{L}$, $\mathcal{D}$ and Z satisfying hypotheses which we will state below.

8.1 First type of system

We consider $4n_i = 4n_0 - i4n \geq n+1 \gg 1$ with $i = 0,1,2,3$. We denote $\mathbb{E}_i = \mathbb{C}^m \times P_c(\omega_0)\Sigma_{4n_i}$ with $i = 0,1,2,3$. Set also $\mathbb{E}_4 = \mathbb{C}^m \times P_c(\omega_0)H^1$ and $j_\epsilon = J_\epsilon^{2n}$. We assume:

(A1) $\mathcal{A}_a$ are real valued functions. $\mathcal{B}_a$ and $\mathcal{C}$ are imaginary valued. $Z_{\underline{j}}$ has values in $\mathbb{C}$ with $Z_{\overline{j}} = \overline{Z}_j$.

(A2) $\mathcal{D}$ has values in Σ_{4n_0} For $G = \mathcal{A}_a, \mathcal{B}_a, \mathcal{C}, \mathcal{D}, Z_{\overline{j}}$ we have $G = G(t,z,f,\varrho(f))$ where $G(t,z,f,\varrho)$ is C^n in t, z, in $f \in \Sigma_{-4n_0}$ and in ρ.

(A3) We have

$$\begin{aligned} &|\mathcal{A}_a| + |Z| + |\mathcal{C}| + \|\mathcal{D}\|_{\Sigma_{4n_0}} \\ &\leq C(|z| + \|f\|_{\Sigma_{-4n_0}} + |\varrho(f)|)(|z| + \|f\|_{\Sigma_{-4n_0}}). \end{aligned} \tag{8.2}$$

(A4) We have either $\mathcal{B}_a = -\frac{\mathrm{i}}{2}v_a - \frac{\mathrm{i}}{2}t dv_a(\mathcal{X}^t)$ for $a = 1,2,3$ or $\mathcal{B}_a \equiv 0$.

The coefficients of Lemma 5.8 satisfy (A1)–(A4) for any choice of n and n_0, by our freedom of choice of space $H^{K,S}$ and by the fact that $H^{K,S} \subset \Sigma_{4n_0}$ for $K > 4n_0$ and $S > 4n_0$. We have:

Proposition 8.1. *The following facts hold.*

(1) *$\exists$ a neighborhood $\mathcal{U}$ of $0 \in \mathbb{E}_0$ defined by $|z| < \varepsilon_0$ and $\|f\|_{H^1} < \varepsilon_0$, s.t. $\forall (\underline{z}, \underline{f}) \in \mathcal{U}$ system (6.1) has exactly one solution $(z(t), f(t)) \in \cap_{i=1}^2 (C^{i-1}([-2,2], \mathbb{E}_i) \cap W^{i-1,\infty}([-2,2], \mathbb{E}_{i-1}))$.*

(2) *Call ϕ^t the flow of (8.1). Then $\partial_t^i \phi^t \in C^i([-2,2], C^n(\mathcal{U}, \mathbb{E}_{2+i}))$ for $i = 0,1$.*

(3) *For $(z^t, f^t) = \phi^t(z,f)$ we have*

$$\begin{aligned} &\|z^t\|_{L^\infty(-2,2)} + \|f^t\|_{L^\infty([-2,2],H^1)} \leq C(|z| + \|f\|_{H^1}), \\ &\|f^t\|_{L^\infty([-2,2],\Sigma_{4n_0})} \leq C(|z| + \|f\|_{\Sigma_{4n_0}}). \end{aligned} \tag{8.3}$$

Proof. We will need to check that we are in the framework and the hypotheses of Sect. 6. Proposition 8.1 is a consequence of Proposition 6.1 if we can prove the hypotheses (1)–(4) in Sect.6. Specifically we need to prove the inequalities in (4) Sect. 6. By Hypotheses (A3)–(A4) and by Lemma 7.5 we see immediately that the analogue of (6.2) is satisfied. (6.3) is a consequence of the following lemma.

Lemma 8.2. *For a fixed constant C and for $C_{|z|+|\varrho(f)|}$ an increasing positive function in $|z|+|\varrho(f)|$, we have:*

$$
\begin{aligned}
&|Z\cdot \overline{z}| \le C|z|(|z|+\|f\|_{\Sigma_{-4n_0}}+|\varrho(f)|)(|z|+\|f\|_{\Sigma_{-4n_0}}),\\
&|(f,\mathcal{D})_{\Sigma_{4n_i}}| \le C\|f\|_{\Sigma_{4n_i}}(|z|+\|f\|_{\Sigma_{-4n_0}}+|\varrho(f)|)(|z|+\|f\|_{\Sigma_{-4n_0}})\,,\\
&|(f,\mathcal{D})_{H^1}| \le C\|f\|_{L^2}(|z|+\|f\|_{\Sigma_{-4n_0}}+|\varrho(f)|)(|z|+\|f\|_{\Sigma_{-4n_0}}),\\
&|(f,J_\epsilon^{2n}\mathcal{L}f)_{\Sigma_{4n_i}}| \le C_{|z|+|\varrho(f)|}\|f\|^2_{\Sigma_{4n_i}}\,,\\
&|(f,J_\epsilon^{2n}\mathcal{L}f)_{H^1}| \le C_{|z|+|\varrho(f)|}\|f\|^2_{H^1}.
\end{aligned}
$$

Proof. The first three inequalities follow immediately from (A2)–(A3). The last two inequalities are an immediate consequence of the following two inequalities for $\widehat{T}=\ 1\!\!1,\partial_a,x_a$ for any $a=1,2,3$: there is a fixed C s.t.

$$
|(f,J_\epsilon^{2n}\widehat{T}f)_{\Sigma_{4n_i}}| \le C\|f\|^2_{\Sigma_{4n_i}}\,,\quad |(f,J_\epsilon^{2n}\widehat{T}f)_{H^1}| \le C\|f\|^2_{H^1}. \tag{8.4}
$$

(8.4) follows from Lemma 7.9.

□

Finally, to finish with the proof of Proposition 6.1 we need to prove that (6.4) is true. But like for (6.3) this too is an easy consequence of (A2)–(A4) and of Lemma 7.9. Hence Proposition 8.1 is proved.

□

8.2 Second type of system

Before setting up the system we notice that for solutions of (8.1) satisfying (A1)–(A4) we have

$$
\begin{aligned}
&\frac{d}{dt}Q(f)=\langle\sigma_1 f|\dot f\rangle=\langle\sigma_1 f|\mathcal{D}\rangle\,,\\
&\frac{d}{dt}\Pi_a(f)=\mathrm{i}\langle\sigma_1\sigma_3\partial_a f|\dot f\rangle=\mathrm{i}\mathcal{B}_b\langle\sigma_1\sigma_3\partial_a f|\sigma_3 x_b f\rangle+\mathrm{i}\langle\sigma_1\sigma_3\partial_a f|\mathcal{D}\rangle\\
&=2\mathrm{i}Q(f)\mathcal{B}_a-\mathrm{i}\langle\sigma_1\sigma_3 f|\partial_a\mathcal{D}\rangle.
\end{aligned} \tag{8.5}
$$

We set $\varrho_0(f):=Q(f)$ and $\varrho_a(f):=\Pi_a(f)$. We consider the system

$$
\begin{aligned}
&\dot\varrho_0=\langle\sigma_1 f|\mathcal{D}\rangle \quad , \quad \dot\varrho_a=2\mathrm{i}\varrho_0\mathcal{B}_a-\mathrm{i}\langle\sigma_1\sigma_3 f|\partial_a\mathcal{D}\rangle,\\
&\dot f=\mathcal{L}f+\mathcal{D} \quad , \quad \dot z=Z.
\end{aligned} \tag{8.6}
$$

We denote $\mathbb{E}_i=\mathbb{R}^4\times\mathbb{C}^m\times P_c(\omega_0)\Sigma_{-4n_{3-i}}$ with $i=0,1,2,3$. We assume:

(B1) same as (A1);

(B2) $\mathcal{D}$ has values in Σ_{4n_3}. For $G = \mathcal{A}_a, \mathcal{B}_a, \mathcal{C}, \mathcal{D}, Z_{\overline{j}}$ we have $G = G(t, z, f, \varrho)$ where $G(t, z, f, \varrho)$ is C^n in t, z, in $f \in \Sigma_{-4n_0}$ and in ρ;

(B3) we have

$$\begin{aligned} &|\mathcal{A}_a| \le C(|z|^2 + \|f\|^2_{\Sigma_{-4n_0}} + |\varrho|)\,, \\ &|Z| + |\mathcal{C}| + \|\mathcal{D}\|_{\Sigma_{4n_0}} \le C(|z| + \|f\|_{\Sigma_{-4n_0}} + |\varrho|)(|z| + \|f\|_{\Sigma_{-4n_0}}); \end{aligned} \tag{8.7}$$

(B4) we have either $|\mathcal{B}_a| \le C|\varrho|$ for $a = 1, 2, 3$ or $\mathcal{B}_a \equiv 0$.

Notice that, starting from $E = \mathcal{A}_a, \mathcal{B}_a, \mathcal{C}, \mathcal{D}, Z_{\overline{j}}$, with $E = E(t, z, f, \varrho(f))$ which satisfy hypotheses (A1)–(A4) in Subsect. 8.1 and substituting $\varrho(f)$ with an external parameter ϱ, we obtain functions satisfying (B1)–(B4).

We have:

Proposition 8.3. *The following facts hold.*

(1) $\exists$ *a neighborhood* $\mathcal{U}$ *of* $0 \in \mathbb{E}_0$ *defined by* $|\varrho| + |z| < \varepsilon_0$ *and* $\|f\|_{\Sigma_{-4n_0}} < \varepsilon_0$, *s.t.* $\forall\, (\underline{z}, \underline{f}) \in \mathcal{U}$ *system* (6.1) *has exactly one solution* $(\varrho(t), z(t), f(t)) \in \cap_{i=1}^{2} (C^{i-1}([-2, 2], \mathbb{E}_i) \cap W^{i-1,\infty}([-2, 2], \mathbb{E}_{i+1}))$.

(2) Call Φ^t *the flow of* (8.6). *Then* $\partial_t^i \Phi^t(y) \in C([-2, 2], C^n(\mathcal{U}, \mathbb{E}_{2+i}))$ *for* $i = 0, 1$.

(3) Set $(z_t, f_t, \varrho_t) := \Phi^t(z, f, \varrho)$. *Then we have for* $i = 0, 3$

$$\|(z^t, \|f^t\|_{\Sigma_{-4n_i}}, \varrho^t)\|_{L^\infty(-2,2)} \le C(|z| + \|f\|_{\Sigma_{-4n_i}} + |\varrho|). \tag{8.8}$$

(4) Call ϕ^t *the flow of* (8.1) *satisfying (A1)–(A4) and suppose that* (8.6) *is the corresponding system substituting* $\varrho(f)$ *with an external parameter* ϱ. *When* $f \in H^1$ *and for* $(z^t, f^t) := \phi^t(z, f)$, *we have* $\Phi^t(\varrho(f), z, f) = (\varrho(f^t), z^t, f^t)$.

Proof. Like in Subsect. 8.1 we can reduce to material in Sect. 6. Specifically, by the arguments of Subsect. 8.1 we can apply Proposition 6.2. This yields the Claims (1)–(3). Claim (4) follows from (8.5) and the uniqueness of solutions in (8.6).

□

8.3 Structure of the Lie transform

Consider system (8.1) such that (A1)–(A4) hold. Consider the corresponding system (8.6) satisfying (B1)–(B4). We denote by $\phi = \phi^1$ the *Lie transform*.

Lemma 8.4. *Set $(z', f') = \phi(z, f)$. Then we have*

$$z' = z + \mathcal{Z} \qquad f'(x) = e^{\sigma_3(\boldsymbol{B}\cdot x+\gamma)}\tau_{\mathbf{A}} f + \mathcal{G}(x) \tag{8.9}$$

with $\tau_{\mathbf{A}} f(x) = f(x - \mathbf{A})$, $\mathbf{A} = -\int_0^1 \mathcal{A}(\tau)d\tau$, $\mathcal{Z} = \int_0^1 Z(\tau)d\tau$, $\boldsymbol{B} = \int_0^1 \mathcal{B}(\tau)d\tau$, $\gamma = \int_0^1 \left(\mathcal{C}(s) + \mathcal{A}_a(s)\int_0^s \mathcal{B}_a(\tau)d\tau\right) ds$ and $\mathcal{G}$ functions of $(z, f, \varrho(f))$

$$\mathcal{G}(x) = \int_0^1 e^{-\sigma_3 \int_s^1 \left(x_a \mathcal{B}_a(\tau)+\mathcal{C}(\tau)-\mathcal{A}_a(\tau)\int_0^\tau \mathcal{B}_a(\tau')d\tau'\right)d\tau} \mathcal{D}(s, x + \int_s^1 \mathcal{A}(\tau)d\tau)ds.$$

We have the following estimates for a fixed constant $C > 0$:

$$\|(z, \|f\|_{\Sigma_{-4n_0}})\|_{L^\infty(-2,2)} \leq C(|z| + \|f\|_{\Sigma_{-4n_0}})\,, \tag{8.10}$$

$$|\mathcal{Z}| + \|\mathcal{G}\|_{\Sigma_{4n_0}} \leq C(|z| + \|f\|_{\Sigma_{-4n_0}} + |\varrho|)(|z| + \|f\|_{\Sigma_{-4n_0}}), \tag{8.11}$$

$$|\boldsymbol{A}| + |\boldsymbol{B}| + |\gamma| \leq C(|z|^2 + \|f\|^2_{\Sigma_{-4n_0}} + |\varrho|). \tag{8.12}$$

We have $\mathcal{Z} = \mathcal{Z}(z, f, \varrho(f))$, $\boldsymbol{A} = \boldsymbol{A}(z, f, \varrho(f))$, $\boldsymbol{B} = \boldsymbol{B}(z, f, \varrho(f))$ and $\gamma = \gamma(z, f, \varrho(f))$, with C^n dependence in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4n_3}$ and $\varrho \in \mathbb{R}^4$. $\mathcal{G} = \mathcal{G}(z, f, \varrho(f))$ has C^n dependence in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4n_3}$ and $\varrho \in \mathbb{R}^4$ with values in Σ_{4n_0-n}. Here $n_0 = n_3 + 2n$ and $4n_3 \geq n + 1$.

Proof. The formulas follow by the use of the integrating factor. The last claim follows from Proposition 8.3. We now consider the estimates (8.10)–(8.12), which improve (8.8). Notice that

$$z^t = z + \mathcal{Z}_t \qquad f^t(x) = e^{\sigma_3(\mathbf{B}_t\cdot x+\gamma_t)}\tau_{\mathbf{A}_t} f + \mathcal{G}_t(x), \tag{8.13}$$

defined similarly to (8.9) but with integrals in $[0, t]$ (resp. $[s, t]$) rather than in $[0, 1]$ (resp. $[s, 1]$). We have by (8.7)–(8.8)

$$|\mathcal{Z}_t| \leq \int_0^t |Z(\tau)|d\tau \leq C(|z| + \|f\|_{\Sigma_{-4n_0}} + |\varrho(f)|) \int_0^t (|z^\tau| + \|f^\tau\|_{\Sigma_{-4n_0}})d\tau.$$

Similarly

$$\int_0^t |\mathcal{C}(\tau)|d\tau \leq C(|z| + \|f\|_{\Sigma_{-4n_0}} + |\varrho(f)|) \int_0^t (|z^\tau| + \|f^\tau\|_{\Sigma_{-4n_0}})d\tau. \tag{8.14}$$

For $\Upsilon = \mathcal{A}, \mathcal{B}$ we have $\int_s^t |\Upsilon(\tau)|d\tau \leq C(|z| + \|f\|_{\Sigma_{-4n_0}} + |\varrho(f)|)^2$. As a consequence we get $\|e^{\sigma_3(\mathbf{B}_t\cdot x+\gamma_t)}\tau_{\mathbf{A}_t} f\|_{\Sigma_{-4n_0}} \leq C\|f\|_{\Sigma_{-4n_0}}$ and

$$\begin{aligned}\|\mathcal{G}_t\|_{\Sigma_{4n_0}} &\leq C \int_0^t \|\mathcal{D}(s, x + \int_s^t \mathcal{A}(\tau)d\tau)\|_{\Sigma_{4n_0}} ds \leq C' \int_0^t \|\mathcal{D}(s, x)\|_{\Sigma_{-4n_0}} ds \\ &\leq C''(|z| + \|f\|_{\Sigma_{-4n_0}} + |\varrho(f)|) \int_0^t (|z^s| + \|f^s\|_{\Sigma_{-4n_0}})ds.\end{aligned}$$

Then (8.10) follows by Gronwall inequality and implies (8.11)–(8.12). $\square$

Remark 8.5. Notice that the theory in Sect 6 entails loss of regularity, in the sense that $\mathbf{A}$ and the other functions are regular in $f \in \Sigma_{-4n_3}$ and not $f \in \Sigma_{-4n_0}$. Since we consider many flows, we have big losses of regularity. Fortunately we consider no more that $2N+2$ transformations and we have a lot of regularity to begin with.

Lemma 8.6. *Consider the system $\dot{f} = (\mathcal{X}^t)_f$ and $\dot{z}_j = (\mathcal{X}^t)_j$. Then the conclusions of Lemma 8.4 continue to hold and we have also*

$$f'(x) = e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot(x-\mathbf{A})+\widetilde{\gamma})} f(x-\mathbf{A}) + \mathcal{G}(x) \tag{8.15}$$

with v' the velocity associated to the $t=1$ vector.

Proof. The starting point is formula (8.9). By Lemma 5.8 we have

$$\mathbf{B} = \int_0^1 \mathcal{B}_a(t)dt = -\frac{\mathrm{i}}{2}\int_0^1 (v_a(t) + t dv_a(\mathcal{X}^t)dt = -\frac{\mathrm{i}}{2}v_a(1).$$

Recalling the $\gamma = \int_0^1 \left(\mathcal{C}(s) + \mathcal{A}_a(s)\int_0^s \mathcal{B}_a(\tau)d\tau\right) ds$ in Lemma 8.4, we have

$$\int_0^1 ds\mathcal{A}_a(s)\int_0^s \mathcal{B}_a(\tau)d\tau = -\frac{\mathrm{i}}{2}\int_0^1 \mathcal{A}_a(s)v_a(s)\,ds =$$
$$\frac{\mathrm{i}}{2}\int_0^1 v_a(s)\frac{d}{ds}\mathbf{A}_a(s)\,ds = \frac{\mathrm{i}}{2}\mathbf{A}_a(1)v_a(1) - \frac{\mathrm{i}}{2}\int_0^1 \mathbf{A}_a(s)\frac{d}{ds}v_a(s)\,ds.$$

We get (8.15) setting $\widetilde{\gamma} := \gamma - \frac{\mathrm{i}}{2}\mathbf{A}\cdot v'$.

□

Lemma 8.7. *z' is C^n in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4n_3}$ and $\varrho \in \mathbb{R}^4$. f' is C^n in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4n_3}$ and $\varrho \in \mathbb{R}^4$ with values in Σ_{-4n_3-n}, where $4n_3 \geq n+1$. $\varrho(f')$ is C^n in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4n_3}$ and $\varrho \in \mathbb{R}^4$.*

Proof. With the notation of Lemmas 8.4 and 8.6, for $\mathcal{Z}$, $\mathbf{A}$, $\widetilde{\gamma} = \gamma - \frac{\mathrm{i}}{2}\mathbf{A}\cdot v'$ and $\mathcal{G}$ we have the result of Lemma 8.4. So for z' the claim follows immediately while for f' is a consequence of formula (8.15), the chain rule and Lemma 7.5. We have

$$\begin{aligned} &Q(f') = Q(f) + 2\langle e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot x+\widetilde{\gamma})}f|\sigma_1\tau_{-\mathbf{A}}\mathcal{G}\rangle + Q(\mathcal{G}) \\ &\Pi_a(f') = \Pi_a(f) - \frac{v'_a}{2}Q(f) + \mathrm{i}\langle e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot x+\widetilde{\gamma})}f|\tau_{-\mathbf{A}}\sigma_1\sigma_3\partial_a\mathcal{G}\rangle + \Pi_a(\mathcal{G}). \end{aligned} \tag{8.16}$$

By Lemma 8.4 we have that $\sigma_1\sigma_3^i e^{\sigma_3(\frac{\mathrm{i}}{2}v'\cdot x-\widetilde{\gamma})}\tau_{-\mathbf{A}}\partial_a^i\mathcal{G}$ for $i=0,1$ is C^n in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4n_3}$ and $\varrho \in \mathbb{R}^4$ with values in Σ_{4n_0-n-i} with $n_0 = n_3 + 2n$. Then for $f \in \Sigma_{-4n_3}$ and by $4n_0 - n - i \geq 4n_3$ it follows that the mixed terms are C^n in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4n_3}$ and $\varrho \in \mathbb{R}^4$. $Q(\mathcal{G})$ and $\Pi_a(\mathcal{G})$ are of the desired type. □

9 Reformulation of (4.4) in the new coordinates

Denote by $\mathcal{F}_t$ the flow of the system (8.1) associated to the field of Lemma 5.8. We set

$$H = K \circ \mathcal{F}_1. \tag{9.1}$$

In the new coordinates (4.4) becomes

$$\mathrm{i}\dot{z}_j = \frac{\partial H}{\partial \overline{z}_j}, \quad \mathrm{i}\dot{f} = \sigma_3\sigma_1\nabla_f H. \tag{9.2}$$

For system (9.2) we prove:

Theorem 9.1. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for $|z(0)| + \|f(0)\|_{H^1} \leq \epsilon < \varepsilon_0$ with $f(0)$ in the Schwartz class, then the corresponding solution of (9.2) is globally defined and there are $f_\pm \in H^1$ with $\|f_\pm\|_{H^1} \leq C\epsilon$ and functions $\widehat{\vartheta} \in C^1(\mathbb{R}, \mathbb{R})$ and $\widehat{D} \in C^1(\mathbb{R}, \mathbb{R}^3)$ such that*

$$\lim_{t\to\pm\infty} \left\| \tau_{\widehat{D}(t)} e^{\mathrm{i}\widehat{\vartheta}(t)\sigma_3} f(t) - e^{\mathrm{i}t\Delta\sigma_3} f_\pm \right\|_{H^1} = 0. \tag{9.3}$$

We have

$$\lim_{t\to\infty} z(t) = 0. \tag{9.4}$$

It is possible to write $f(t,x) = A(t,x) + \widetilde{f}(t,x)$ with $|A(t,x)| \leq C_N(t)\langle x\rangle^{-N}$ for any N, with $\lim_{t\to\infty} C_N(t) = 0$ and such that for any admissible pair (p,q), i.e. (1.6), we have $\|\widetilde{f}\|_{L^p_t(\mathbb{R}, W^{1,q}_x)} \leq C\epsilon$.

We now move to the proof of Theorem 9.1. First of all we remark that in the sequel we need that the hamiltonians and the Lie transforms be sufficiently regular. The amount of regularity needed depends on $N = N_1$. We will consider a total of $2(N+1)$ Lie transforms, considering both the implementation of Darboux theorem and the Birkhoff normal forms. We need to end up with a final hamiltonian which is at least C^1. We can make sure that all the hamiltonians and Lie transforms are sufficiently regular by picking $H^{K,S}$ with $K \gg 2N$ and $S \gg 2N$ in Lemmas 4.5 and 5.8.

The first step in the proof of Theorem 9.1. is a preliminary discussion of $H = K \circ \mathcal{F}_1$. In Sections 10–12 we implement the method of Birkhoff normal forms, looking for other coordinates. Finally we will settle in the right system of coordinates and in Sect. 13 we will finally prove estimates.

Lemma 9.2. *Fix a large number $M \in \mathbb{N}$ with $M \gg 2N$. We have the expansion*

$$H = \psi(\varrho(f)) + H_2^{(1)} + \mathcal{R}^{(1)} \tag{9.5}$$

where $\psi(\varrho)$ is C^M in ϱ and where:

(1) We have for $\ell = 1$

$$H_2^{(\ell)} = \sum_{\substack{|\mu+\nu|=2 \\ \lambda^0\cdot(\mu-\nu)=0}} a_{\mu\nu}^{(\ell)}(\varrho(f)) z^\mu \overline{z}^\nu + \frac{1}{2}\langle \sigma_3 \mathcal{H}_{\omega_0} f | \sigma_1 f\rangle. \tag{9.6}$$

(2) We have $\mathcal{R}^{(1)} = \widetilde{\mathcal{R}^{(1)}} + \widetilde{\mathcal{R}^{(2)}}$, with $\widetilde{\mathcal{R}^{(1)}} =$

$$= \sum_{\substack{|\mu+\nu|=2 \\ \lambda^0\cdot(\mu-\nu)\neq 0}} a_{\mu\nu}^{(1)}(\varrho(f)) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=1} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 G_{\mu\nu}(\varrho(f)) | f\rangle \tag{9.7}$$

$$\begin{aligned}\widetilde{\mathcal{R}^{(2)}} &= \sum_{|\mu+\nu|=3} z^\mu \overline{z}^\nu a_{\mu\nu}(z,f,\varrho(f)) + \sum_{|\mu+\nu|=2} z^\mu \overline{z}^\nu \langle G_{\mu\nu}(z,f,\varrho(f)) | \sigma_3\sigma_1 f\rangle \\ &+ \sum_{d=2}^{4} \langle B_d(z,f,\varrho(f)) | f^d\rangle + \int_{\mathbb{R}^3} B_5(x,z,f,f(x),\varrho(f)) f^5(x) dx \\ &+ \widehat{\mathcal{R}}_2^{(1)}(z,f,\varrho(f)) + E_P(f).\end{aligned}$$

with $B_2(0,0,0)=0$ and where, both here and in Theorem 12.1 later, by $f^d(x)$ we schematically represent $d-$products of components of f.

(3) At $\varrho(f)=0$ with $\ell=1$

$$\begin{aligned}&a_{\mu\nu}^{(\ell)}(0) = 0 \text{ for } |\mu+\nu| = 2 \text{ with } (\mu,\nu)\neq(\delta_j,\delta_j) \text{ for all } j, \\ &a_{\delta_j\delta_j}^{(\ell)}(0) = \lambda_j(\omega_0), \text{ where } \delta_j = (\delta_{1j},...,\delta_{mj}), \\ &G_{\mu\nu}(0) = 0 \text{ for } |\mu+\nu| = 1\end{aligned} \tag{9.8}$$

These $a_{\mu\nu}^{(\ell)}(\varrho)$ and $G_{\mu\nu}(x,\varrho)$ are C^M in all variables with $G_{\mu\nu}(\cdot,\varrho) \in C^M(\mathbb{R}^4, \Sigma_{4M}(\mathbb{R}^3,\mathbb{C}^2))$.

(4) For a small neighborhood U of $(0,0,0)$ in $\mathbb{C}^m\times\Sigma_{-4M}\times\mathbb{R}^4$, we have $a_{\mu\nu}(z,\varrho)\in C^M(\mathrm{U},\mathbb{C})$.

(5) $G_{\mu\nu}(\cdot,z,\varrho)\in C^M(\mathrm{U},\Sigma_{4M}(\mathbb{R}^3,\mathbb{C}^2))$.

(6) $B_d(\cdot,z,f,\varrho)\in C^M(\mathrm{U},\Sigma_{4M}(\mathbb{R}^3,B((\mathbb{C}^2)^{\otimes d},\mathbb{C})))$, for $2\le d\le 4$.

(7) Let ${}^t\eta = (\zeta,\overline{\zeta})$ for $\zeta\in\mathbb{C}$. Then for $B_5(\cdot,z,f,\eta,\varrho)$ we have

$$\text{for } |l|\le M \ , \ \|\nabla^l_{z,\overline{z},f,\zeta,\overline{\zeta},\varrho} B_5(z,f,\eta,\varrho)\|_{\Sigma_{4M}(\mathbb{R}^3,B((\mathbb{C}^2)^{\otimes 5},\mathbb{C})}\le C_l.$$

(8) We have for all indexes and for $\ell=1$

$$a_{\mu\nu}^{(\ell)} = \overline{a}_{\nu\mu}^{(\ell)}, \quad a_{\mu\nu} = \overline{a}_{\nu\mu}, \quad G_{\mu\nu} = -\sigma_1\overline{G}_{\nu\mu}. \tag{9.9}$$

(9)

$$
\begin{aligned}
&\widehat{\mathcal{R}}_2^{(1)} \in C^M(\mathrm{U},\mathbb{R}),\\
&|\widehat{\mathcal{R}}_2^{(1)}(z,f,\varrho)| \le C(|z|+|\varrho|+\|f\|_{\Sigma_{-4M}})\|f\|^2_{\Sigma_{-4M}};
\end{aligned}
\tag{9.10}
$$

Proof. We consider the notation of Lemma 8.6. Thanks to Lemma 8.4 and by the freedom of choice of $H^{K,S}$ in Lemmas 4.5 and 5.8, we can assume that $\mathbf{A}$ and $v' \in C^{\widetilde{M}}(\widetilde{\mathrm{U}},\mathbb{R}^3)$, $\mathcal{G} \in C^{\widetilde{M}}(\widetilde{\mathrm{U}},\Sigma_{4\widetilde{M}})$ and $\widetilde{\gamma} \in C^{\widetilde{M}}(\widetilde{\mathrm{U}},\mathrm{i}\mathbb{R})$, with $\widetilde{\mathrm{U}}$ a neighborhood of the origin in the space $\varrho(f)=\varrho\in\mathbb{R}^4$, $z\in\mathbb{C}^m$ and $f\in\Sigma_{-4\widetilde{M}}$ and with $\widetilde{M}\gg M$. Having in mind (8.16), we notice that $e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot x+\widetilde{\gamma})}\partial_a^i\tau_{-\mathbf{A}}\mathcal{G}\in C^{\widehat{M}}(\widetilde{\mathrm{U}},\Sigma_{4\widetilde{M}-4\widehat{M}-i})$ for $i=0,1$. Set $F:=e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot x+\widetilde{\gamma})}\tau_{-\mathbf{A}}\mathcal{G}$. Then we have, for $\varrho=\varrho(f)$ and $G(t)=F(tz,tf,\varrho)$,

$$
\begin{aligned}
&\langle f|F\rangle = \langle f|(z_j\partial_j+\overline{z}_j\partial_{\overline{j}})F(0,0,\varrho)\rangle\\
&+\langle f|\partial_f F(0,0,\varrho)f\rangle+\frac{1}{2}\int_0^1\langle f|\frac{d^2}{dt^2}G(t)\rangle dt.
\end{aligned}
\tag{9.11}
$$

The first term in the right is like the $z^\mu\overline{z}^\nu\langle\sigma_1\sigma_3G_{\mu\nu}(\varrho(f))|f\rangle$ in (9.7) with $G_{\mu\nu}(\varrho)\in C^{\widehat{M}-1}(\mathbb{R}^4,\Sigma_{4\widetilde{M}-4\widehat{M}})$. By taking appropriate $\widetilde{M}$ and $\widehat{M}$, the conditions in the statement will hold. The second term in the rhs of (9.11) is like $\widehat{\mathcal{R}}_2^{(1)}(z,f,\varrho)$, satisfying (9.10) for appropriate choices of $\widetilde{M}$ and $\widehat{M}$ by (8.11). The last term in (9.11) is higher order, and again is like $\widehat{\mathcal{R}}_2^{(1)}$ or can be absorbed in $\widetilde{\mathcal{R}}^{(2)}$. Similar expansions hold for $Q(\mathcal{G})$ and analogous terms on the rhs of the second equality in (8.16). Expanding like in (8.16) we obtain

$$
\begin{aligned}
&(z')^\mu(\overline{z}')^\nu\langle\sigma_1\sigma_3\underline{G}_{\mu\nu}(\varrho(f'))|f'\rangle=\\
&(z+Z)^\mu(\overline{z}+\overline{Z})^\nu\langle\sigma_1\sigma_3\underline{G}_{\mu\nu}(\varrho(f'))|\tau_{\mathbf{A}}e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot x+\widetilde{\gamma})}f+\mathcal{G}\rangle.
\end{aligned}
\tag{9.12}
$$

Analogous formulas hold for $\langle\underline{B}_d(z',\varrho(f'))|(f')^d\rangle$ for $d=2,3,4$ and $E_P(f')$. With Taylor expansions similar to (9.11) we get in an elementary fashion that (9.12) expands into terms falling in one of the cases in the statement. For $d=5$, $0\le j\le 5$ and with some exponentials in absorbed $\underline{B}$, schematically, we have terms like

$$
\int_{\mathbb{R}^3}\widehat{\underline{B}}_5(x+\mathbf{A},z',e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot x+\widetilde{\gamma})}f(x)+\mathcal{G}(x+\mathbf{A}),\varrho(f'))f^j(x)\mathcal{G}^{5-j}(x+\mathbf{A})dx
$$

which by Taylor expansion and by the fact that $\widetilde{M}\gg M$ can be absorbed in $\widetilde{\mathcal{R}^{(2)}}$. We next look at

$$
\langle\sigma_3\mathcal{H}_{\omega_0}f'|\sigma_1f'\rangle=\langle(-\Delta+\omega_0)f'|\sigma_1f'\rangle+\langle\sigma_3V_{\omega_0}f'|\sigma_1f'\rangle.
$$

We have

$$
\begin{aligned}
&\langle(-\Delta+\omega_0)f'|\sigma_1f'\rangle=\langle(-\Delta+\omega_0)f|\sigma_1f\rangle+\frac{v^2}{2}Q(f)+2v_a\Pi_a(f)\\
&+\langle(-\Delta+\omega_0)\mathcal{G}|\sigma_1\mathcal{G}\rangle+2\langle(-\Delta+\omega_0)\mathcal{G}|\sigma_1\tau_{\mathbf{A}}e^{\sigma_3(-\frac{\mathrm{i}}{2}v'\cdot x+\widetilde{\gamma})}f\rangle.
\end{aligned}
\tag{9.13}
$$

In (9.13) the last line can be treated as above and absorbed in the $\mathcal{R}^{(1)}$ while the last two terms of the first line go in part in $\psi(\varrho(f))$ and in part in $\mathcal{R}^{(1)}$ by $v = 2\Pi(R)/Q(U)$. The term $\langle \sigma_3 V_{\omega_0} f', \sigma_1 f' \rangle$ can be expanded as the sum of $\langle \sigma_3 V_{\omega_0} f, \sigma_1 f \rangle$ plus a reminder term treating it as (9.12).

$\square$

10 Normal forms and homological equation

We set $\mathcal{H} = \mathcal{H}_{\omega_0} P_c(\mathcal{H}_{\omega_0})$. Consider $\mathbb{C}$ valued functions $a^{(\ell)}_{\mu\nu}(\varrho)$ such that $a^{(\ell)}_{\nu\mu} \equiv \overline{a^{(\ell)}_{\mu\nu}}$. We assume that $a^{(\ell)}_{\mu\nu} \in C^{k_0}(U, \mathbb{C})$ for $k_0 \in \mathbb{N}$ a fixed number and U a neighborhood of 0 in $\mathbb{R}^4$. Then we set

$$H_2^{(\ell)}(\varrho) := \sum_{\substack{|\mu+\nu|=2 \\ \lambda(0)\cdot(\mu-\nu)=0}} a^{(\ell)}_{\mu\nu}(\varrho) z^\mu \overline{z}^\nu + \frac{1}{2}\langle \sigma_3 \mathcal{H}_{\omega_0} f | \sigma_1 f \rangle. \tag{10.1}$$

$$\lambda_j^{(\ell)}(\varrho) := a^{(\ell)}_{\delta_j \delta_j}(\varrho), \quad \lambda^{(\ell)} = (\lambda_1^{(\ell)}, \cdots, \lambda_m^{(\ell)}). \tag{10.2}$$

We assume $\lambda_j^{(\ell)}(0) = \lambda_j(\omega_0)$ and $a^{(\ell)}_{\mu\nu}(0) = 0$ if $(\mu, \nu) \neq (\delta_j, \delta_j)$ for all j, with δ_j defined in (9.8).

Definition 10.1. A function $Z(z, f, \varrho)$ is in normal form if it is a sum

$$Z = Z_0 + Z_1 \tag{10.3}$$

where Z_0 and Z_1 are finite sums of the following type:

$$Z_1 = \sum_{|\lambda(\varrho)\cdot(\nu-\mu)|>\omega_0} z^\mu \overline{z}^\nu \langle \sigma_1 \sigma_3 G_{\mu\nu}(\varrho) | f \rangle \tag{10.4}$$

with $G_{\mu\nu}(x, \omega, \varrho) \in C^{k_0}(U, \Sigma_{k_1})$ for fixed $k_0 \in \mathbb{N}$;

$$Z_0 = \sum_{\lambda(0)\cdot(\mu-\nu)=0} a_{\mu\nu}(\varrho) z^\mu \overline{z}^\nu \tag{10.5}$$

and $a_{\mu\nu}(\varrho) \in C^{k_0}(U, \mathbb{C})$. We will always assume the symmetries (4.12). $\square$

For $G = G(x)$, by elementary computations we have

$$\begin{aligned}
&\frac{1}{2}\{\langle \sigma_3 \mathcal{H} f | \sigma_1 f \rangle, \langle \sigma_1 \sigma_3 G, f \rangle\} = -\mathrm{i}\langle f | \sigma_1 \sigma_3 \mathcal{H} G \rangle, \\
&\frac{1}{2}\{\langle \sigma_3 \mathcal{H} f | \sigma_1 f \rangle, Q(f)\} = \mathrm{i}\langle \mathcal{H} f | \sigma_1 f \rangle = -\mathrm{i}\langle \beta'(\phi^2)\phi^2 \sigma_3 f | f \rangle, \\
&\frac{1}{2}\{\langle \sigma_3 \mathcal{H} f | \sigma_1 f \rangle, \Pi_a(f)\} = \langle \sigma_3 \mathcal{H} f | \sigma_1 \partial_a f \rangle = -\frac{1}{2}\langle \sigma_3 (\partial_a V_{\omega_0}) f | \sigma_1 f \rangle, \\
&\{f, Q(f)\} = -\mathrm{i} P_c(\omega_0)\sigma_3 f, \quad \{f, \Pi_a(f)\} = P_c(\omega_0)\partial_a f.
\end{aligned} \tag{10.6}$$

We now discuss the homological equations. We start by assuming that ϱ is an external parameter

Lemma 10.2. *We consider $\chi = \chi(b, B)$ with*

$$\chi(b, B) = \sum_{|\mu+\nu|=M_0+1} b_{\mu\nu} z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 B_{\mu\nu} | f\rangle \tag{10.7}$$

for $b_{\mu\nu} \in \mathbb{C}$ and $B_{\mu\nu} \in \Sigma_{2k_1}$ with $k_1 \in \mathbb{N}$. Here we interpret the polynomial χ as a function with parameters $b = (b_{\mu\nu})$ and $B = (B_{\mu\nu})$. Denote by X_{2k_1} the space of the pairs (b, B). Let us also consider given polynomials with $K = K(\varrho)$ and $\widetilde{K} = K(\varrho, b, B)$ where:

$$K(\varrho) := \sum_{|\mu+\nu|=M_0+1} k_{\mu\nu}(\varrho) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 K_{\mu\nu}(\varrho) | f\rangle, \tag{10.8}$$

with $k_{\mu\nu}(\varrho) \in C^{k_0}(U, \mathbb{C})$ and $K_{\mu\nu}(\varrho) \in C^{k_0}(U, \Sigma_{2k_1} \cap P_c(\omega_0)L^2)$ for U a neighborhood of 0 in $\mathbb{R}^4$; let

$$\begin{aligned} \widetilde{K}(\varrho, b, B) &:= \sum_{|\mu+\nu|=M_0+1} \widetilde{k}_{\mu\nu}(\varrho, b, B) z^\mu \overline{z}^\nu \\ &+ \sum_{i=0}^{1}\sum_{a=1}^{3}\sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 \partial_a^i K^i_{a\mu\nu}(\varrho, b, B) | f\rangle, \end{aligned} \tag{10.9}$$

with $\widetilde{k}_{\mu\nu} \in C^{k_0}(U \times X_{2k_1}, \mathbb{C})$ and $\widetilde{K}^i_{a\mu\nu} \in C^{k_0}(U \times X_{2k_1}, \Sigma_{2k_1} \cap P_c(\omega_0)L^2)$. Suppose also that the sums (10.8) and (10.9) do not contain terms in normal form and that $\widetilde{K}(0, b, B) = 0$. Then there exists a neighborhood $V \subseteq U$ of 0 in $\mathbb{R}^4$ and a unique choice of functions $(b(\varrho), B(\varrho)) \in C^{k_0}(V, X_{2k_1})$ such that for $\chi(\varrho) = \chi(b(\varrho), B(\varrho))$, $\widetilde{K}(\varrho) = \widetilde{K}(\varrho, b(\varrho), B(\varrho))$ we have

$$\{\chi(\varrho), H_2(\varrho)\} = K(\varrho) + \widetilde{K}(\varrho) + Z(\varrho) \tag{10.10}$$

where $Z(\varrho)$ is in normal form and homogeneous of degree $M_0 + 1$ in $(z, \overline{z}, f)$. If the coefficients of K satisfy the symmetries in (4.12), the same is true for χ.

Proof. Summing on repeated indexes, we get

$$\begin{aligned} &\{H_2, \chi\} = \mathrm{i}\lambda(\varrho) \cdot (\mu - \nu) z^\mu \overline{z}^\nu b_{\mu\nu} \\ &+ \mathrm{i}\langle f | \sigma_1\sigma_3 (\lambda(\varrho) \cdot (\mu - \nu) - \mathcal{H}) B_{\mu\nu}\rangle + \widehat{K}(\varrho, b, B), \end{aligned} \tag{10.11}$$

where, with an abuse of notation,

$$\widehat{K}(\varrho, b, B) := \sum_{\mu\nu\mu'\nu'} a^{(\ell)}_{\mu\nu}(\varrho) \left(b_{\mu'\nu'} + \langle \sigma_1\sigma_3 B_{\mu'\nu'} | f\rangle\right) \{z^\mu \overline{z}^\nu, z^{\mu'} \overline{z}^{\nu'}\}, \tag{10.12}$$

with the sum only on $|\mu + \nu| = 2$ with $(\mu, \nu) \neq (\delta_j, \delta_j)$ for all j. $\widehat{K}(\varrho, b, B)$ is 0 for $\varrho = 0$ and is a homogeneous polynomial of the same type of the above

ones. Denote by $\widehat{Z}(\varrho, b, B)$ the sum of its monomials in normal form and set $\mathbf{K} := \widetilde{K} + \widehat{K} - \widehat{Z}$. We look at

$$\begin{aligned} &\mathrm{i}\lambda(\varrho)\cdot(\mu-\nu)z^\mu\overline{z}^\nu b_{\mu\nu} + z^\mu\overline{z}^\nu \mathrm{i}\langle f|\sigma_1\sigma_3(\lambda(\varrho)\cdot(\mu-\nu)-\mathcal{H})B_{\mu\nu}\rangle \\ &+\mathbf{K}\ (\varrho, b, B) + K(\varrho) = 0 \end{aligned} \tag{10.13}$$

that is at

$$\begin{aligned} &k_{\mu\nu}(\varrho) + \mathbf{k}_{\mu\nu}(\varrho, b, B) + \mathrm{i}b_{\mu\nu}\lambda(\varrho)\cdot(\mu-\nu) = 0 \\ &K_{\mu\nu}(\varrho) + \mathbf{K}_{\mu\nu}(\varrho, b, B) - \mathrm{i}(\mathcal{H}-\lambda(\varrho)\cdot(\mu-\nu))B_{\mu\nu} = 0, \end{aligned} \tag{10.14}$$

with $\mathbf{k}_{\mu\nu}$ and $\mathbf{K}_{\mu\nu}$ the coefficients of $\mathbf{K}$. Notice that by $\mathbf{k}_{\mu\nu}(0, b, B) = 0$ and $\mathbf{K}_{\mu\nu}(0, b, B) = 0$, for $\varrho = 0$ there is a unique solution $(b, B) \in X_{2k_1}$ given by

$$b_{\mu\nu} = \frac{\mathrm{i}k_{\mu\nu}(0)}{\lambda(0)\cdot(\mu-\nu)}\,, \quad B_{\mu\nu}(0) = -\mathrm{i}R_{\mathcal{H}}(\lambda(0)\cdot(\mu-\nu))K_{\mu\nu}(0). \tag{10.15}$$

Notice that for $i = 0, 1$ we have $R_{\mathcal{H}}(\zeta)P_c(\omega_0)\circ\partial_a^i \in B(\Sigma_{2l}, \Sigma_{2l})$ for any $l \in \mathbb{Z}$, $\zeta \notin \sigma_e(\mathcal{H})$ and $a \in \{1, 2, 3\}$. The case $i = 0$ can be proved by induction over l using material in Sect. 7. The case $i = 1$ holds for $\mathcal{H}_0 := \sigma_2(-\Delta+\omega)$. Finally, these facts and the resolvent identity yield the case $i = 1$ for $\mathcal{H}$. Then Lemma 10.2 is a consequence of the implicit function theorem.

□

Substituting $\varrho = \varrho(f)$ we obtain what follows:

Lemma 10.3. *Set $K_1 = K(\varrho(f))$, $\widetilde{K}_1 = \widetilde{K}(\varrho(f), b(\varrho(f)), B(\varrho(f)))$ and $\chi_1 = \chi(\varrho(f))$. Then we have*

$$\{\chi_1, H_2\} = K_1 + \widetilde{K}_1 + Z_1 + L_1 \tag{10.16}$$

where Z_1 is in normal form and homogeneous of degree $M_0 + 1$ in $(z, \overline{z}, f)$ and

$$L_1 = \langle V_j(\varrho(f))f|f\rangle\widetilde{\chi}_j + \langle T_j f|f\rangle\widehat{\chi}_j, \tag{10.17}$$

where: $V_j(\varrho) \in C^{k_0-1}(U, \Sigma_{2k_1-1})$, $T_j \in B(\Sigma_{-2k_1}, \Sigma_l)$ for all l; $\widetilde{\chi}_j$ and $\widehat{\chi}_j$ polynomials like χ_1, with monomials of no smaller degree and with coefficients in C^{k_0-1} in ϱ.

Proof. By direct computation (10.16) holds with

$$\begin{aligned} L_1 = &\,\mathrm{i}\langle\beta'(\phi^2)\phi^2\sigma_3 f|f\rangle\partial_{Q(f)}\chi_1 + \frac{1}{2}\langle\sigma_3(\partial_a V_{\omega_0})f|\sigma_1 f\rangle\partial_{\Pi_a(f)}\chi_1 + \\ &z^\mu\overline{z}^\nu\partial_{\varrho_i}a_{\mu\nu}^{(\ell)}\left(\partial_{\varrho_j}\chi_1\{\varrho_i(f), \varrho_j(f)\} + z^{\mu'}\overline{z}^{\nu'}\langle\sigma_1\sigma_3 B_{\mu'\nu'}, \{\varrho_i(f), f\}\rangle\right), \end{aligned} \tag{10.18}$$

with $\varrho_0(f) = Q(f)$ and $\varrho_a(f) = \Pi_a(f)$ for $a = 1, 2, 3$. We have

$$\begin{aligned} &\{Q(f), \Pi_a(f)\} = \mathrm{i}\langle\sigma_1 f|P_c(\omega_0)\partial_a f\rangle = \mathrm{i}\langle\sigma_1 f|\partial_a P_d(\omega_0)f\rangle; \\ &\langle\sigma_1\sigma_3 B_{\mu\nu}|\{Q(f), f\}\rangle = \langle B_{\mu\nu}|P_c(\omega_0)\sigma_3 f\rangle; \\ &\{\Pi_b(f), \Pi_a(f)\} = \mathrm{i}\langle\sigma_1\sigma_3\partial_b f|P_c(\omega_0)\partial_a f\rangle = -\mathrm{i}\langle\sigma_1\sigma_3\partial_b f|P_d(\omega_0)\partial_a f\rangle; \\ &\langle\sigma_1\sigma_3 B_{\mu\nu}|\{\Pi_a(f), f\}\rangle = \langle B_{\mu\nu}|P_c(\omega_0)\partial_a f\rangle \\ &= -\langle\partial_a B_{\mu\nu}|f\rangle - \langle B_{\mu\nu}|P_d(\omega_0)\partial_a f\rangle. \end{aligned} \tag{10.19}$$

(10.18)– (10.19) yield the properties of L_1. $\square$

11 Canonical transformations

We consider functions χ

$$\chi = \sum_{|\mu+\nu|=M_0+1} b_{\mu\nu}(\varrho(f)) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 B_{\mu\nu}(\varrho(f)) | f\rangle. \tag{11.1}$$

We assume $b_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \mathbb{C})$ and $B_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \Sigma_{4k_1}(\mathbb{R}^3, \mathbb{C}^2))$ satisfying the symmetries in (4.12). Here k_0 and k_1 are fixed, very large and in $\mathbb{N}$ and $k_1 \gg k_0$. We will always assume $B_{\mu\nu} \in L^2_c(\mathcal{H}_{\omega_0})$. We want to consider the flow ϕ^t associated to the Hamiltonian vector field X_χ at time $t = 1$ and use it to change coordinates.

Our first step consists in setting up the hamiltonian system associated to χ. It is a quasilinear symmetric hyperbolic system.

Lemma 11.1. *Consider χ as in* (11.1) *satisfying the symmetries in* (4.12)*. Summing on repeated indexes, the following holds.*

$$\{f, \chi\} = \mathcal{L}f + \mathcal{D} \ , \quad \{z_j, \chi\} = Z_j \ , \quad \mathcal{L} := \mathcal{A}_a \partial_a + \mathcal{C}\sigma_3 \tag{11.2}$$

where the coefficients are given by the following formulas:

$$\begin{aligned} &\mathcal{A}_a = \partial_{\Pi_a(f)}\chi \quad , \quad \mathcal{C} = -\mathrm{i}\partial_{Q(f)}\chi \quad , \quad Z_j = -\mathrm{i}\partial_{\overline{z}_j}\chi, \\ &\mathcal{D} = -\mathrm{i} z^\mu \overline{z}^\nu B_{\mu\nu}(\varrho(f)) - P_d(\omega_0)\mathcal{L}f. \end{aligned} \tag{11.3}$$

The coefficients can be thought as dependent on $(z, f, \varrho(f))$. If we substitute $\varrho(f)$ with an independent variable $\varrho \in \mathbb{R}^4$, then we have $\mathcal{A}_a \in C^{k_0-1}(\mathfrak{V}, \mathbb{R})$, $\mathrm{i}\mathcal{B}_a, \mathrm{i}\mathcal{C} \in C^{k_0-1}(\mathfrak{V}, \mathbb{R})$ and $\mathcal{D} \in C^{k_0-1}(\mathfrak{V}, \Sigma_{4k_1})$, with $\mathfrak{V}$ a neighborhood of the origin in $\mathbb{C}^m \times \Sigma_{8k_0-4k_1} \times \mathbb{R}^4$.

The following inequalities hold:

$$\begin{aligned} &|\mathcal{A}| + |\mathcal{C}| \le C|z|^{M_0}(|z| + \|f\|_{\Sigma_{-4k_1}}) \\ &|Z| + \|\mathcal{D}\|_{\Sigma_{4k_1}} \le C|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1}}). \end{aligned} \tag{11.4}$$

Proof. (11.2)–(11.3) follow from a simple computation. The rest follows by Subsect. 8.3 for $n_0 = k_1$ and $n = k_0$. $\square$

Lemma 11.1 assures us that we are within the framework of Sec. 8 and that the Lie transform $\phi = \phi^1$ associated to the following system is well defined:

$$\dot{f} = \{f, \chi\} \, , \quad \dot{z} = \{z, \chi\}. \tag{11.5}$$

In particular, we have:

Lemma 11.2. *Suppose k_1 is sufficiently large. Set $(z^t, f^t) = \phi^t(z, f)$. Then we have*

$$z^t = z + \mathcal{Z}_t\,, \qquad f^t = e^{\sigma_3 \gamma_t} \tau_{\mathbf{A}_t} f + \mathcal{G}_t \tag{11.6}$$

with $\tau_{\mathbf{A}_t} f(x) = f(x - \mathbf{A}_t)$, $\mathbf{A}_t = -\int_0^t \mathcal{A}(\tau) d\tau$, $\mathcal{Z}_t = \int_0^t Z(\tau) d\tau$, $\gamma_t = \int_0^t \mathcal{C}(s) ds$ and

$$\mathcal{G}_t(x) = \textstyle\int_0^t e^{-\sigma_3 \int_s^t \mathcal{C}(\tau) d\tau} \mathcal{D}(s, x + \int_s^t \mathcal{A}(\tau) d\tau) ds.$$

We have $\boldsymbol{A}_t = \boldsymbol{A}_t(z, f, \varrho(f))$, and $\gamma_t = \gamma_t(z, f, \varrho(f))$, with C^{k_0-1} dependence in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$. The same statement holds for $\mathcal{Z}_t = \mathcal{Z}_t(z, f, \varrho(f))$ resp. $\mathcal{G}_t = \mathcal{G}_t(z, f, \varrho(f))$ with values in $\mathbb{C}^m$ resp. $\Sigma_{4k_1-k_0}$.
The f^t has C^{k_0-1} dependence in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$ with values in $\Sigma_{9k_0-4k_1}$.
The $\varrho(f^t)$ has C^{k_0-1} dependence in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho(f) \in \mathbb{R}^4$ with values in $\mathbb{R}^4$.
There is a fixed constant C such that

$$|\mathcal{Z}_t| + \|\mathcal{G}_t\|_{\Sigma_{4k_1}} \le C|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1}}), \tag{11.7}$$

$$|\mathbf{A}_t| + |\gamma_t| \le C|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1}})^2. \tag{11.8}$$

Proof. This is the analogue of Lemmas 8.4 and 8.6 . ∎

We will set $\phi = \phi^1$, $(z', f') = \phi(z, f)$ and we will drop the subindex t for $t = 1$, that is $\mathbf{A} = \mathbf{A}_t$ etc.

Lemma 11.3. *In the above notation we have*

$$|Q(f') - Q(f)| \le C|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1}})^2, \tag{11.9}$$

$$|\Pi(f') - \Pi(f)| \le C|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1+1}})^2 \tag{11.10}$$

for a fixed C dependent on $\|(b_{\mu\nu}, B_{\mu\nu})\|_{C^1(\mathrm{U}_4)}$ with $\mathrm{U}_4 \subset \mathbb{R}^4$ a preassigned neighborhood of the origin.

Proof. We have

$$Q(f') = Q(f) + \int_0^1 \{Q(f), \chi\} \circ \phi^t dt, \tag{11.11}$$

with

$$\{Q(f), \chi\} = \{Q(f), \Pi_a(f)\} \partial_{\Pi_a(f)} \chi + z^\mu \overline{z}^\nu \langle \sigma_1 \sigma_3 B_{\mu\nu}(\varrho(f)) | \{Q(f), f\} \rangle.$$

We have $|\{Q(f), \chi\}| \le C(|b| + \|B\|_{\Sigma_{4k_1}})|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1}})^2$ for a fixed C dependent on $\|(b_{\mu\nu}, B_{\mu\nu})\|_{C^1}$ by the formulas in (10.19). The integral in (11.11) has the same upper bound by Lemma 11.2, in particular by (11.6) and

inequalities (11.7)–(11.8). This proves (11.9). For $\Pi_a(f')$ we have a similar argument by the following formulas and estimates:

$$\{\Pi_b(f), \Pi_a(f)\} = \mathrm{i}\langle \sigma_1\sigma_3\partial_b f | P_c(\omega_0)\partial_a f\rangle = -\mathrm{i}\langle \sigma_1\sigma_3\partial_b f | P_d(\omega_0)\partial_a f\rangle;$$

$|\langle \sigma_1\sigma_3\partial_b f | P_d(\omega_0)\partial_a f\rangle| \le C_\ell \|f\|^2_{\Sigma_{-\ell}}$ and $|\langle B_{\mu\nu}|P_d(\omega_0)\partial_a f\rangle| \le C_\ell \|B\|_{\Sigma_{-\ell}} \|f\|_{\Sigma_{-\ell}}$ for any ℓ. We have $|\langle \partial_a B_{\mu\nu}|f\rangle| \le C\|B\|_{\Sigma_{4k_1}} \|f\|_{\Sigma_{-4k_1+1}}$ by $\|\partial_a B\|_{\Sigma_{4k_1-1}} \le c\|B\|_{\Sigma_{4k_1}}$, see Lemma 7.5. Thanks to (10.19) and Lemma 11.2 these inequalities yield (11.10). □

The information in Lemma 11.2 is not sufficiently precise for our purposes. Let us consider for any fixed $(z(0), f(0))$ the system

$$\begin{aligned} &\dot g = -\mathrm{i}\zeta^\mu \zeta^\nu B_{\mu\nu}(\varrho(f(0))\,, \\ &\dot\zeta_j = -\mathrm{i}\nu_j \frac{\zeta^\mu\zeta^\nu}{\overline{\zeta}_j} \left(b_{\mu\nu}(\varrho(f(0)) + \langle \sigma_1\sigma_3 B_{\mu\nu}(\varrho(f(0))|f\rangle\right) \\ &g(0) = f(0)\,, \quad \zeta(0) = z(0). \end{aligned} \tag{11.12}$$

Notice that well posedness, regularity of the flow, and smooth dependence on the coefficients $(b(\varrho(f(0)), B(\varrho(f(0)))$ fall within the scope of the theory of ordinary equations. Denote ϕ_0^t the flow of (11.12). In particular for $(\zeta^t, g^t) = \phi_0^t(\zeta, g)$ we have

$$\begin{aligned} &\zeta^t = \zeta + \mathbf{Z}_t(\zeta, g, b(\varrho(f)), B(\varrho(f))) \\ &g^t = g + \mathbf{G}_t(\zeta, g, b(\varrho(f)), B(\varrho(f))), \end{aligned} \tag{11.13}$$

with $\mathbf{Z}_t(\zeta, g, b, B)$ (resp. $\mathbf{G}_t(\zeta, g, b, B)$) with C^∞ dependence on t, $\zeta \in \mathbb{C}^m$ and $g \in \Sigma_{-4k_1}$, and (b, B), with values in $\mathbb{C}^m$ (resp. $\Sigma_{4k_1}(\mathbb{R}^3, \mathbb{C}^2)$) and with

$$\mathbf{Z}_t(\zeta, g, 0, 0) \equiv 0, \quad \mathbf{G}_t(\zeta, g, 0, 0) \equiv 0. \tag{11.14}$$

Furthermore $\mathbf{Z}_t$ resp. $\mathbf{G}_t$ satisfy uniformly in t the same bounds (11.7) of $\mathcal{Z}$ resp. $\mathcal{G}$.

We compare the solutions of (11.5) with those of (11.12). Denote $(z', f') = \phi^1(z, f)$ and $(\zeta', g') = \phi_0^1(z, f)$.

Lemma 11.4. *For a C like in Lemma 11.3 we have for any $j \le k_1$*

$$|z' - \zeta'| + \|f' - g'\|_{\Sigma_{-4j-1}} \le C(|z| + \|f\|_{\Sigma_{-4j+1}})^{M_0+1}. \tag{11.15}$$

Proof. Set $\mathbf{M}(t) = |z(t)| + |\zeta(t)| + \|f(t)\|_{\Sigma_{1-4j}} + \|g(t)\|_{\Sigma_{1-4j}}$. We have

$$\dot f - \dot g = \mathrm{i}\zeta^\mu \overline{\zeta}^\nu B_{\mu\nu}(\varrho(f(0))) - \mathrm{i}z^\mu \overline{z}^\nu B_{\mu\nu}(\varrho(f)) - P_d(\omega_0)\mathcal{L}f + (\mathcal{A}_a\partial_a + \mathcal{C}\sigma_3)f.$$

The rhs has Σ_{-4j} norm bounded by

$$\begin{aligned} &|\zeta^\mu\overline{\zeta}^\nu - z^\mu\overline{z}^\nu|\,\|B_{\mu\nu}(\varrho(f(0)))\|_{\Sigma_{-4j}} + |z^\mu\overline{z}^\nu|\|B_{\mu\nu}(\varrho(f)) - B_{\mu\nu}(\varrho(f(0)))\|_{\Sigma_{-4j}} \\ &+ \|P_d(\omega_0)\mathcal{L}f\|_{\Sigma_{-4j}} + \|(\mathcal{A}_a\partial_a + \mathcal{C}\sigma_3)f\|_{\Sigma_{-4j}}. \end{aligned}$$

Then

$$\|f(t)-g(t)\|_{\Sigma_{-4j}} \le C\int_0^t \mathbf{M}^{M_0-1}(t)|z(\tau)-\zeta(\tau)|d\tau + C\int_0^t \mathbf{M}^{M_0+2}(t)d\tau.$$

Similarly

$$\begin{aligned}&|z(t)-\zeta(t)| \le C\int_0^t \mathbf{M}^{M_0-1}(t)(|z(\tau)-\zeta(\tau)| + \|f(\tau)-g(\tau)\|_{\Sigma_{-4j}})d\tau\\ &+ C\int_0^t \mathbf{M}^{M_0+1}(t)d\tau.\end{aligned}$$

Then the statement follows from Gronwall inequality since $\mathbf{M}(t) \le C\mathbf{M}(0)$. For instance, $|z(t)| + \|f(t)\|_{\Sigma_{1-4j}} \le C(|z(0)| + \|f(0)\|_{\Sigma_{1-4j}})$ follows by formulas (11.6) and by inequalities (11.7)–(11.8). □

Lemma 11.5. *In the above notation of Lemma 11.5 we have*

$$|Q(f')-Q(g')| \le C(|z| + \|f\|_{\Sigma_{-4k_1+1}})^{M_0+2}, \tag{11.16}$$

$$|\Pi(f')-\Pi(g')| \le C(|z| + \|f\|_{\Sigma_{-4k_1+3}})^{M_0+2}. \tag{11.17}$$

Proof. We have

$$\begin{aligned}&\frac{d}{dt}(\Pi_b(f)-\Pi_b(g)) = \mathrm{i}\zeta^\mu\overline{\zeta}^\nu\langle B_{\mu\nu}(\varrho(f(0)))|\sigma_3\sigma_1\partial_b g\rangle\\ &+\mathrm{i}z^\mu\overline{z}^\nu\langle B_{\mu\nu}(\varrho(f))|\sigma_3\sigma_1\partial_b f\rangle - \langle P_d(\omega_0)\mathcal{L}f|\sigma_3\sigma_1\partial_b f\rangle.\end{aligned}$$

The right hand side can be bounded above by the rhs of (11.17) computed at time t. Integrating, using the fact that $(z,f) = (\zeta,g)$ at $t=0$ and by (11.6) and (11.7)–(11.8) we get (11.17). The proof of (11.16) is similar.

□

12 Birkhoff normal forms

Our goal in this section is to prove the following result.

Theorem 12.1. *For any integer $2 \le \ell \le 2N+1$ there are a $\delta_0 > 0$ and $M \gg N$ large such that in the subset of Σ_{4M} defined by $|z| + \|f\|_{H^1} < \delta_0$ is defined a canonical transformation $\mathcal{T}_r$ which is differentiable as a map with values in Σ_1 and whose image contains a similar subset of Σ_{4M} defined by $|z| + \|f\|_{H^1} < \delta_0'$, s.t.*

$$H^{(\ell)} := K\circ\mathcal{T}_\ell = \psi(\varrho(f)) + H_2^{(\ell)} + Z^{(\ell)} + \mathcal{R}^{(\ell)}, \tag{12.1}$$

with $\psi(\varrho(f))$ the same of (9.5) and where:

(i) $H_2^{(\ell)} = H_2^{(2)}$ for $\ell \ge 2$, is of the form (4.9) where $a_{\mu\nu}^{(\ell)}$ satisfy (9.8)–(9.9);

(ii) $Z^{(\ell)}$ *is in normal form, with monomials of degree* $\le \ell$ *whose coefficients satisfy* (4.12)*;*

(iii) *we have* $\mathcal{T}_\ell = \phi_\ell \circ ... \circ \phi_1$*, with each* ϕ_j *a Lie transformation associated to a function* (11.1) *with* $M_0 = j$*;*

(iv) *we have* $\mathcal{R}^{(\ell)} = \sum_{d=0}^{6} \mathcal{R}_d^{(\ell)}$ *with the following properties (for* $k_2(\ell) \ll k_3(\ell) \ll M$ *pairs of appropriate large numbers with* $k_j(\ell+1) \ll k_j(\ell)$ *for all* ℓ *and for* $j = 2, 3$*):*

(iv.0) *we have with* $|\partial_\varrho^l a_{\mu\nu}^{(\ell)}(\varrho)| \le C_l$ *for* $|l| \le k_2(\ell)$*,*

$$\mathcal{R}_0^{(\ell)} = \sum_{|\mu+\nu|=\ell+1} z^\mu \overline{z}^\nu a_{\mu\nu}^{(\ell)}(\varrho(f)) :$$

(iv.1) *we have with* $\|\partial_\varrho^l G_{\mu\nu}^{(\ell)}(\varrho)(\cdot)\|_{\Sigma_{4k_3(\ell)}(\mathbb{R}^3,\mathbb{C}^2)} \le C_l$ *for* $|l| \le k_2(\ell)$*,*

$$\mathcal{R}_1^{(\ell)} = \sum_{|\mu+\nu|=\ell} z^\mu \overline{z}^\nu \langle \sigma_1 \sigma_3 G_{\mu\nu}^{(\ell)}(\varrho(f)) | f \rangle \text{ with}$$

(iv.2–5) *in* $\mathcal{U}$ *we have for* $2 \le d \le 5$ *and for* $\eta^T = (\zeta, \overline{\zeta})$ *with* $\zeta \in \mathbb{C}$*,*

$$\mathcal{R}_d^{(\ell)} = \int_{\mathbb{R}^3} F_d^{(\ell)}(x, z, f, f(x), \varrho(f)) f^d(x) dx + \widehat{\mathcal{R}}_d^{(\ell)},$$

with for $|l| \le k_2(\ell)$

$$\|\partial^l_{z,\overline{z},\zeta,\overline{\zeta},f,\varrho} F_d^{(\ell)}(\cdot, z, f, \eta, \varrho)\|_{\Sigma_{4k_3(\ell)}(\mathbb{R}^3, B((\mathbb{C}^2)^{\otimes d}, \mathbb{C})} \le C_l, \tag{12.2}$$

with $F_2^{(\ell)}(x, 0, 0, 0, 0) = 0$ *and with* $\widetilde{\mathcal{R}}_d^{(\ell)}(z, f, \varrho(f))$ *s.t.*

$$\begin{aligned} &\widehat{\mathcal{R}}_d^{(\ell)} \in C^{k_2(\ell)}(\mathcal{U} \times \mathbb{R}, \mathbb{R}), \quad |\widehat{\mathcal{R}}_d^{(\ell)}(z, f, \varrho)| \le C\|f\|^d_{\Sigma_{-4k_3(\ell)}}, \\ &|\widehat{\mathcal{R}}_2^{(\ell)}(z, f, \varrho)| \le C(|z| + |\varrho| + \|f\|_{\Sigma_{-4k_3(\ell)}})\|f\|^2_{\Sigma_{-4k_3(\ell)}}; \end{aligned} \tag{12.3}$$

(iv.6) $\mathcal{R}_6^{(\ell)} = \int_{\mathbb{R}^3} B(|f(x)|^2/2) dx$.

12.1 Pullback of multilinear forms

The method of Birkhoff normal forms is implemented using the flows of auxiliary hamiltonians χ like in (11.1). In particular, in we will assume for the moment that the degree is $M_0 + 1$ and that $b_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \mathbb{C})$ and $B_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \Sigma_{4k_1}(\mathbb{R}^3, \mathbb{C}^2))$. In the proof of Theorem 12.1, χ needs to solve a *homological equation.* In this section we look at pullbacks of the various terms of the hamiltonian by means of the Lie transform associated to χ. In general these

terms are pulled back into other reminder terms which are less of regular. By this we mean both that their coefficients are less than C^{k_0} and with values in some Σ_{4k} with $k < k_1$. In general this loss of regularity is harmless. However we have to make sure that the terms which enter in the homological equation of χ, which is used to find a useful χ, have same regularity of χ. It is at this stage that we use the associated simplified system (11.12). We will consider now a number of technical lemmas.

Lemma 12.2. *Let $F = F(z, f, \varrho)$ be C^{k_0} in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4k_1}$ and $\varrho \in \mathbb{R}^4$ with values in $\Sigma_{4k_1}(\mathbb{R}^3, B^2(\mathbb{C}, \mathbb{C}))$. For $M_0 = 1$ we assume $F(0, 0, 0) = 0$. Then*

$$\begin{aligned} \langle F(z', f', \varrho(f'))|\mathcal{G}^2\rangle = &\sum_{|\mu+\nu|=M_0+1} k_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f)))z^\mu \overline{z}^\nu \\ + &\sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 K_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f)))|f\rangle + \boldsymbol{R}, \end{aligned} \tag{12.4}$$

where the following holds.

(i) $k_{\mu\nu}(0, b, B) = 0$ and $K_{\mu\nu}(0, b, B) = 0$; $k_{\mu\nu}(\varrho, b, B) \in \mathbb{C}$ and $K_{\mu\nu}(\varrho, b, B) \in \Sigma_{4k_1}$ are C^{k_0-1} in ϱ, in $b_{\mu\nu} \in \mathbb{C}$ and in $B_{\mu\nu} \in \Sigma_{4k_1}$.

(ii) $\boldsymbol{R}$ is a sum of terms of the form $\mathcal{R}^{(M_0+1)}$, that is like in the statement of Theorem 12.1, with $k_3 = k_1 - 2k_0$ and $k_2 = k_0 - M_0 - 3$.

If $M_0 > 1$ formula (12.4) holds with only $\boldsymbol{R}$ in the rhs.

Proof. We have for $\varrho = \varrho(f)$ and $\varrho' = \varrho(f')$ and $\delta\varrho = \varrho' - \varrho$

$$\langle F(z', f', \varrho')|\mathcal{G}^2\rangle = \langle F(z', f', \varrho)|\mathcal{G}^2\rangle + \int_0^1 \langle \partial_\varrho F(z', f', \varrho + t\delta\varrho)|\mathcal{G}^2\rangle dt \cdot \delta\varrho. \tag{12.5}$$

By Lemma 11.2 the second term in the rhs is C^{k_0-1} in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$. Furthermore $\delta\varrho$ satisfies $|\delta\varrho| \le C|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1+1}})^2$ by Lemma 11.3. By $k_0 = k_2 + M_0 + 3$ we can write the second term in the rhs of (12.5) as $\mathcal{R}_0^{(M_0+1)} + \mathcal{R}_1^{(M_0+1)} + \widehat{\mathcal{R}}_2^{(M_0+1)}$, just by performing an appropriate and partial Taylor expansion. If $M_0 > 1$ the same result holds for the first term in the rhs of (12.5). Let now $M_0 = 1$. By Lemmas 11.2 and 8.7 it is C^{k_0-1} in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$ and can be expressed as $\langle F(0, 0, \varrho)|\mathcal{G}^2\rangle$ plus a term which is like the above ones and can be absorbed in $\mathbf{R}$. Writing $F(\varrho) = F(0, 0, \varrho)$, we have succinctly

$$\langle F(\varrho)|\mathcal{G}^2\rangle = \langle F(\varrho)|\mathbf{G}_1^2\rangle - 2\langle F(\varrho)|(\mathbf{G}_1 - \mathcal{G})\mathbf{G}_1\rangle + \langle F(\varrho)|(\mathbf{G}_1 - \mathcal{G})^2\rangle, \tag{12.6}$$

with $\mathbf{G}_1$ from the associated system, see (11.13). By Lemma 11.4 the rhs is C^{k_0-1} in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$. We have

$$\langle F(\varrho)|(\mathbf{G}_1 - \mathcal{G})\mathbf{G}_1\rangle = \langle F(\varrho)|(g' - f')\mathbf{G}_1\rangle + \langle F(\varrho)|(e^{\sigma_3\gamma}\tau_{\mathbf{A}} f - f)\mathbf{G}_1\rangle.$$

Then

$$\begin{aligned}
&|\langle F(\varrho)|(g'-f')\mathbf{G}_1\rangle| \le \|F\|_{\Sigma_{4k_1}}\|\mathbf{G}_1\|_{\Sigma_{4k_1}}\|g'-f'\|_{\Sigma_{4k_1}} \\
&\le C\|F\|_{\Sigma_{4k_1}}(|z|+\|f\|_{\Sigma_{1-4k_1}})^3.
\end{aligned}$$

Similarly

$$\begin{aligned}
&|\langle F(\varrho)|(e^{\sigma_3\gamma}\tau_{\mathbf{A}}f-f)\mathbf{G}_1\rangle| \le \|F\|_{\Sigma_{4k_1}}\|\mathbf{G}_1\|_{\Sigma_{4k_1}}\|(e^{\sigma_3\gamma}\tau_{\mathbf{A}}f-f)\|_{\Sigma_{4k_1}} \\
&\le C\|F\|_{\Sigma_{4k_1}}(|z|+\|f\|_{\Sigma_{1-4k_1}})^3.
\end{aligned}$$

Hence the second term in the rhs of (12.6) can be absorbed in $\mathbf{R}$. Similar reasoning applies to the third term in the rhs of (12.6). We finally show that the first term in the rhs of (12.6) yields the first two terms in the rhs of (12.4) plus a term which can be absorbed in $\mathbf{R}$. We know that $\mathbf{G}_1 = \mathbf{G}_1(z, f, b_{\mu\nu}, B_{\mu\nu})$ is C^∞ in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4k_1}$ and $(b_{\mu\nu}, B_{\mu\nu})$, with values in $\Sigma_{4k_1}(\mathbb{R}^3, \mathbb{C}^2)$ with $\|\mathbf{G}_1\|_{\Sigma_{4k_1}} = O(|z| + \|f\|_{\Sigma_{-4k_1}})$. We can consider

$$\begin{aligned}
\langle F(\varrho)|\mathbf{G}_1^2\rangle &= \sum_{|\mu+\nu|=2} h_{\mu\nu}(\varrho,b,B)z^\mu\overline{z}^\nu \\
&+\sum_{|\mu+\nu|=1} z^\mu\overline{z}^\nu\langle\sigma_1\sigma_3H_{\mu\nu}(\varrho,b,B)|f\rangle+\widetilde{\mathcal{R}},
\end{aligned}\tag{12.7}$$

with $\widetilde{\mathcal{R}} = O(|z| + \|f\|_{\Sigma_{-4k_1}})^3$ and C^{k_0} in z, f, ϱ, and with

$$\begin{aligned}
h_{\mu\nu}(\varrho,b,B) &:= \frac{1}{\mu!\nu!}\partial_z^\mu\partial_{\overline{z}}^\nu\langle F(\varrho)|\mathbf{G}_1^2\rangle_{|(0,0,\varrho)}\ , \\
H_{\mu\nu}(\varrho,b,B) &:= \sigma_3\sigma_1\partial_z^\mu\partial_{\overline{z}}^\nu\nabla_f\langle F(\varrho)|\mathbf{G}_1^2\rangle_{|(0,0,\varrho)}\ .
\end{aligned}$$

$\widetilde{\mathcal{R}}$ can be absorbed in $\mathbf{R}$. The polynomial in (12.7) is like the one in the statement because of the hypothesis $F(\varrho) = F(0,0,\varrho) = 0$ when $\varrho = 0$ if $M_0 = 1$. ☐

Lemma 12.3. *Let $F = F(z, f, \varrho)$ with the same properties as in Lemma 12.2. Then, for a rhs which satisfies the same properties stated in Lemma 12.2 but with $k_3 = k_1 - 3k_0$ and $k_2 = k_0 - M_0 - 4$,*

$$\begin{aligned}
\langle F(z',f',\varrho(f'))|\mathcal{G}f'\rangle &= \sum_{|\mu+\nu|=M_0+1} k_{\mu\nu}(\varrho(f),b(\varrho(f)),B(\varrho(f)))z^\mu\overline{z}^\nu \\
&+\sum_{|\mu+\nu|=M_0} z^\mu\overline{z}^\nu\langle\sigma_1\sigma_3K_{\mu\nu}(\varrho(f),b(\varrho(f)),B(\varrho(f)))|f\rangle+\boldsymbol{R}.
\end{aligned}\tag{12.8}$$

Proof. We have

$$\begin{aligned}
&\langle F(z',f',\varrho(f'))|\mathcal{G}f'\rangle = \langle F(z',f',\varrho(f'))|\mathcal{G}f\rangle + \\
&\langle F(z',f',\varrho(f'))|\mathcal{G}(e^{\sigma_3\gamma}\tau_{\mathbf{A}}-1)f\rangle + \langle F(z',f',\varrho(f'))|\mathcal{G}^2\rangle.
\end{aligned}\tag{12.9}$$

The third term in the rhs is like in Lemma 12.3. The second can be absorbed in $\mathbf{R}$ by (11.8). We focus on first term in the rhs of (12.9). By (11.7) we have

$$\begin{aligned} \mathcal{G}(x) = & \int_0^1 \mathcal{D}(s,x)ds \\ & + \int_0^1 \left(e^{-\sigma_3 \int_s^t \mathcal{C}(\tau)d\tau} - 1\right) \mathcal{D}(s, x + \int_s^t \mathcal{A}(\tau)d\tau)ds \\ & + \int_0^1 \left((\mathcal{D}(s, x + \int_s^t \mathcal{A}(\tau)d\tau) - \mathcal{D}(s,x)\right) ds. \end{aligned} \tag{12.10}$$

The last two lines are C^{k_0-1} in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$ with values in $\Sigma_{4k_1-4k_0}$ where they have norm smaller than $C(|z| + \|f\|_{\Sigma_{8k_0-4k_1}})^3$. This implies that when we substitute (12.10) in the first term in the rhs of (12.9), the last two lines of (12.10) can be absorbed in $\mathbf{R}$. When we substitute (11.3), the first term in the rhs of (12.10) is equal to what follows:

$$-\mathrm{i}\int_0^1 (z^\mu \overline{z}^\nu B_{\mu\nu}(\varrho(f))) \circ \phi^t dt - \int_0^1 P_d(\omega_0)(\mathcal{L}f) \circ \phi^t dt. \tag{12.11}$$

By Lemma 11.1 the second term in (12.11) is C^{k_0-1} in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$ with values in Σ_l for any l and with norm $O(|z| + \|f\|_{\Sigma_{4k_0-4k_1}})^{M_0+2}$. The corresponding terms in (12.9) can be then absorbed in $\mathbf{R}$. The first term in (12.11) can be written as

$$-\mathrm{i}B_{\mu\nu}(\varrho(f)) \int_0^1 (z^\mu \overline{z}^\nu) \circ \phi_0^t dt \tag{12.12}$$

plus an error term which can be then absorbed in $\mathbf{R}$ since it is C^{k_0-1} in $z \in \mathbb{C}^m$, $f \in \Sigma_{8k_0-4k_1}$ and $\varrho \in \mathbb{R}^4$ with values in Σ_{4k_1} and with norm smaller than $C(|z| + \|f\|_{\Sigma_{8k_0-4k_1}})^{M_0+2}$. By (11.13) we have that (12.12) is

$$-\mathrm{i}B_{\mu\nu}(\varrho(f))z^\mu \overline{z}^\nu - \mathrm{i}B_{\mu\nu}(\varrho(f))F_{\mu\nu}(z, f, \varrho(f), b(\varrho(f)), B(\varrho(f)), \tag{12.13}$$

with $F_{\mu\nu}(z, f, \varrho, b, B) \in \mathbb{C}$, C^∞ in $\zeta \in \mathbb{C}^m$, $f \in \Sigma_{-4k_1}$, $\varrho \in \mathbb{R}^4$ and in (b, B), Furthermore $|F_{\mu\nu}| \le C|z|^{M_0-1}(|z| + \|f\|_{\Sigma_{-4k_1}})$. The contribution in (12.9) is

$$(z^\mu \overline{z}^\nu + F_{\mu\nu})\langle F(z', f', \varrho(f'))|B_{\mu\nu}(\varrho(f))f\rangle. \tag{12.14}$$

Then proceeding as in Lemma 12.2 we get a contribution like in the rhs of (12.8). $\square$

Lemma 12.4. *Let $\widehat{\mathcal{R}}_d = \widehat{\mathcal{R}}_d(z, f, \varrho)$ be C^{k_0+2} in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4k_1}$ and $\varrho \in \mathbb{R}^4$ with values in $\mathbb{R}$ for $d \ge 2$. Suppose that the following inequalities hold:*

$$\begin{aligned} & |\widehat{\mathcal{R}}_d(z, f, \varrho)| \le C\|f\|^d_{\Sigma_{-4k_1}}, \\ & |\widehat{\mathcal{R}}_2(z, f, \varrho)| \le C(|z| + |\varrho| + \|f\|_{\Sigma_{-4k_1}})\|f\|^2_{\Sigma_{-4k_1}}. \end{aligned} \tag{12.15}$$

Then for any $d = 2, ..., 5$ and for any pair (k_2, k_3) there are $k_1(d)$ and $k_0(d)$ such that for $k_1 \ge k_1(d)$, $k_0 \ge k_0(d)$ and $k_1 \ge Ck_0$ for some fixed large constant

C, then the following occurs: if $d \geq 3$, $\widehat{\mathcal{R}}_d(z', f', \varrho(f'))$ is of the form $\mathcal{R}^{(M_0+1)}$; if $d = 2$ we have

$$\widehat{\mathcal{R}}_2(z', f', \varrho(f')) = \widehat{\mathcal{R}}_2(z, f, \varrho(f)) + \textit{rhs like } (12.4). \tag{12.16}$$

Proof. We will only sketch the case $d = 2$, the others being similar. Schematically $\widehat{\mathcal{R}}_2 = Gf^2$, with $G(z, f, \varrho) \in C^{k_0}$ with values in $B^2(\Sigma_{-4k_1}, \mathbb{C})$ and with $\|G(z, f, \varrho)\|_{B^2(\Sigma_{-4k_1}, \mathbb{C})} = O(|z| + |\varrho| + \|f\|_{\Sigma_{-4k_1}})$. We have

$$\begin{aligned} &G(z', f', \varrho(f'))(f')^2 = G(z', f', \varrho(f'))\mathcal{G}^2 + \\ &2G(z', f', \varrho(f'))\mathcal{G}e^{\mathrm{i}\sigma_3\gamma}\tau_{\mathbf{A}}f + G(z', f', \varrho(f'))\left(e^{\mathrm{i}\sigma_3\gamma}\tau_{\mathbf{A}}f\right)^2. \end{aligned} \tag{12.17}$$

The last term can be easily see to be of the form $\widehat{\mathcal{R}}_2^{(M_0+1)}$. The first two terms can be treated like in Lemmas 12.2 and 12.3. For example, for a $\widetilde{\mathbf{R}}$ which can be absorbed in $\mathbf{R}$, we have

$$G(z', f', \varrho(f'))\mathcal{G}^2 = G(0, 0, \varrho(f))\mathbf{G}_1^2 + \widetilde{\mathbf{R}}.$$

This follows from the same argument used for (12.6) The first term in the rhs yields a term like the rhs of (12.4) exactly by the same argument used for (12.7), this time using the second inequality in (12.15)

$\square$

Lemma 12.5. *Let*

$$\psi = \sum_{|\mu+\nu|=M+1} d_{\mu\nu}(\varrho(f))z^\mu\overline{z}^\nu + \sum_{|\mu+\nu|=M} z^\mu\overline{z}^\nu\langle\sigma_1\sigma_3 D_{\mu\nu}(\varrho(f))|f\rangle$$

be another polynomial like χ, in particular with $d_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \mathbb{C})$ and $D_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \Sigma_{4k_1}(\mathbb{R}^3, \mathbb{C}^2))$ with $M_1 \geq 2$. Then, for any pair (k_2, k_3) there are k_{10} and k_{00} such that for $k_1 \geq k_{10}$, $k_0 \geq k_{00}$ and $k_1 \geq Ck_0$ for some fixed large constant C, then we have, for an $\boldsymbol{R}$ like in Lemma 12.2,

$$\psi \circ \phi - \psi = \boldsymbol{R}. \tag{12.18}$$

Proof. We have $\psi \circ \phi = \psi + \int_0^1 \{\psi, \chi\} \circ \phi^t dt$. By elementary computation using (10.6) we get

$$\begin{aligned} &\{\psi, \chi\} = \mathrm{i}\partial_{\overline{j}}\psi\partial_j\chi - \mathrm{i}\partial_{\overline{j}}\chi\partial_j\psi + \mathrm{i}z^{\alpha+\mu}\overline{z}^{\beta+\nu}\langle D_{\alpha\beta}|\sigma_1\sigma_3 B_{\mu\nu}\rangle + \\ &\partial_{\varrho_i}\psi z^\mu\overline{z}^\nu\langle\sigma_1\sigma_3 B_{\mu\nu}|\{\varrho_i(f), f\}\rangle - \partial_{\varrho_i}\chi z^\alpha\overline{z}^\beta\langle\sigma_1\sigma_3 D_{\alpha\beta}|\{\varrho_i(f), f\}\rangle \\ &+ (\partial_{\varrho_i}\psi\partial_{\varrho_j}\chi - \partial_{\varrho_j}\psi\partial_{\varrho_i}\chi)\{\varrho_i(f), \varrho_j(f)\}. \end{aligned} \tag{12.19}$$

By the formulas (10.19) all the terms the last line are of the form $\widehat{\mathcal{R}}_2$ like in Lemma 12.4. Then by Lemma 12.4 we have

$$\widehat{\mathcal{R}}_2 \circ \phi^t = \widehat{\mathcal{R}}_2 + \text{rhs like } (12.4)_t, \tag{12.20}$$

where the last term depends on t. Integrating in t we eliminate this dependence. Hence the terms from the last line of (12.19) are absorbed in $\mathbf{R}$. A similar conclusion holds for the terms from the second line of (12.19), this time using the last line of (10.6). Finally, the first line of (12.19) is of the form

$$h = \sum_{|\mu+\nu|=M+M_0} h_{\mu\nu}(\varrho) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=M+M_0-1} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 H_{\mu\nu}(\varrho)|f\rangle$$

with coefficients $h_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \mathbb{C})$ and $H_{\mu\nu} \in C^{k_0}(\mathbb{R}^4, \Sigma_{4k_1}(\mathbb{R}^3, \mathbb{C}^2))$. Then $\int_0^1 h \circ \phi^t dt$ can be absorbed in $\mathbf{R}$. $\square$

Lemma 12.6. *Let $F = F(z, f, \eta, \varrho)$ be C^{k_0} in $z \in \mathbb{C}^m$, $f \in \Sigma_{-4k_1}$, $\eta \in \mathbb{C}^2$ and $\varrho \in \mathbb{R}^4$ with values in $\Sigma_{4k_1}(\mathbb{R}^3, B^d(\mathbb{C}, \mathbb{C}))$. If $d = 2$ let $F(0,0,0,0,0) = 0$. Set*

$$R(z, f, \varrho) = \int_{\mathbb{R}^3} F(x, z, f, f(x), \varrho) f^d(x) dx.$$

Then, for any $d = 2, ..., 5$ and for any pair (k_2, k_3) there are $k_1(d)$ and $k_0(d)$ such that for $k_1 \ge k_1(d)$, $k_0 \ge k_0(d)$, $k_1 \ge Ck_0$ for some fixed large constant C and for $k_{\mu\nu}$, $K_{\mu\nu}$ and $\boldsymbol{R}$ like in Lemma 12.2, we have

$$\begin{aligned} R \circ \phi = &\sum_{|\mu+\nu|=M_0+1} k_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f))) z^\mu \overline{z}^\nu \\ + &\sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 K_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f)))|f\rangle + \boldsymbol{R}. \end{aligned} \tag{12.21}$$

If $d > 2$ formula (12.21) holds with only $\boldsymbol{R}$ in the rhs.

Proof. The rhs of (12.21) can be written as a sum of terms of the form for $0 \le i \le d$

$$\int_{\mathbb{R}^3} F(x, z', f', e^{\sigma_3\gamma}\tau_{\mathbf{A}} f(x) + \mathcal{G}(x), \varrho(f')) \mathcal{G}^{d-i}(x) (e^{\sigma_3\gamma}\tau_{\mathbf{A}} f)^i dx. \tag{12.22}$$

The terms with $i \ge 2$ satisfy the statement if we change variable of integration. In particular for $i = 2$ we have $F = 0$ at $(z, f) = (0, 0)$, where we exploit $\mathcal{G} = 0$. If $d \ge 3$, then the (12.22) can be incorporated in $\mathbf{R}$. We then consider the case $d = 2$ and $i = 0, 1$. For $i = 0$, (12.22) is a sum of the form

$$\begin{aligned} &\int_{\mathbb{R}^3} F(x, z', f', \mathcal{G}(x), \varrho(f')) \mathcal{G}^2(x) dx + \\ &\int_{\mathbb{R}^3} G(x + \mathbf{A}, z', f', e^{\sigma_3\gamma} f(x), \tau_{-\mathbf{A}}\mathcal{G}(x), \varrho(f')) \mathcal{G}^2(x + \mathbf{A}) e^{\sigma_3\gamma} f(x) dx, \end{aligned}$$

where the second line can be incorporated in $\mathbf{R}$ and the first line is like Lemma 12.2. If in (12.22) we have $d = 2$ and $i = 1$ we have an expression of the form

$$\int_{\mathbb{R}^3} F(x, z', f', \mathcal{G}(x), \varrho(f'))\mathcal{G}(x)e^{\sigma_3\gamma}\tau_{\mathbf{A}} f(x)dx$$
$$+ \int_{\mathbb{R}^3} G(x+\mathbf{A}, z', f', e^{\sigma_3\gamma} f(x), \mathcal{G}(x+\mathbf{A}), \varrho(f'))(e^{\sigma_3\gamma} f(x))^2 dx.$$

The second line is absorbed in $\mathbf{R}$. The first line is

$$\begin{aligned} &\int_{\mathbb{R}^3} F(x+\mathbf{A}, z', f', \tau_{-\mathbf{A}}\mathcal{G}(x), \varrho(f'))\mathcal{G}(x+\mathbf{A})e^{\sigma_3\gamma} f(x)dx \\ &= \int_{\mathbb{R}^3} F(x, z', f', 0, \varrho(f'))\mathcal{G}(x) f(x)dx + \widetilde{\mathbf{R}} \end{aligned} \tag{12.23}$$

where $\widetilde{\mathbf{R}}$ can be absorbed in $\mathbf{R}$. So we can apply Lemma 12.3.

$\square$

12.2 Proof of Theorem 12.1: the step $\ell = 2$

At this stage our goal is to obtain a hamiltonian similar to H but with $\widetilde{\mathcal{R}^{(1)}} = 0$ in (9.7). In Lemma 9.2 we can assume M arbitrarily large. We consider a polynomial χ, initially unknown, like in (10.7) with $M_0 = 1$ and with k_0 and k_1 arbitrarily large with $1 \ll k_0 \ll k_1 \ll M$. We choose $2N \ll k_2(2) \ll k_3(2) \ll k_1 - k_0$ with $k_2(2)$ as large as needed. We write

$$H \circ \phi = (\psi + H_2^{(1)} + \widetilde{\mathcal{R}^{(1)}} + \widetilde{\mathcal{R}^{(2)}}) \circ \phi, \tag{12.24}$$

for ϕ the Lie transform of χ. We have

$$H_2^{(1)} \circ \phi = H_2^{(1)} + \int_0^1 \{H_2^{(1)}, \chi\} \circ \phi^t dt.$$

By the computations in Sect. 10 we have schematically, for $\ell = 1$,

$$\begin{aligned} &\{H_2^{(\ell)}, \chi\} = \mathrm{i} \sum_{|\mu+\nu|=\ell+1} \lambda^{(\ell)}(\varrho) \cdot (\mu - \nu) z^\mu \overline{z}^\nu b_{\mu\nu} \\ &+ \mathrm{i} \sum_{|\mu+\nu|=\ell} z^\mu \overline{z}^\nu \langle f | \sigma_1\sigma_3(\lambda^{(\ell)}(\varrho) \cdot (\mu - \nu) - \mathcal{H}) B_{\mu\nu} \rangle \\ &+ \sum_{\substack{|\alpha+\beta|=2 \\ (\alpha,\beta)\neq(\delta_j,\delta_j)\,\forall j}} a_{\alpha\beta}^{(\ell)}(\varrho) \sum_{\mu\nu} (b_{\mu\nu} + \langle \sigma_1\sigma_3 B_{\mu\nu} | f \rangle) \{z^\alpha \overline{z}^\beta, z^\mu \overline{z}^\nu\} + L, \end{aligned}$$

with L like (10.18) with χ_1 replaced by χ. Then, in the notation of (10.17) and for ϕ_0^t the flow of the simplified system (11.12), we have

$$
\begin{aligned}
& H_2^{(1)} \circ \phi - H_2^{(1)} = \\
& \sum_{|\mu+\nu|=2} \lambda^{(1)}(\varrho(f)) \cdot (\mu-\nu) b_{\mu\nu}(\rho(f)) \int_0^1 (z^\mu \overline{z}^\nu) \circ \phi_0^t dt \\
& + \sum_{|\mu+\nu|=1} \langle \sigma_1\sigma_3 B_{\mu\nu}(\varrho(f)), (\lambda(\varrho(f)) \cdot (\mu-\nu) + \mathcal{H}) \int_0^1 (z^\mu \overline{z}^\nu f) \circ \phi_0^t dt \rangle \\
& + \sum_{\substack{|\alpha+\beta|=2 \\ (\alpha,\beta)\neq(\delta_j,\delta_j)\,\forall j}} a_{\alpha\beta}^{(1)}(\varrho(f)) \big[\sum_{|\mu+\nu|=2} b_{\mu\nu}(\varrho(f)) \int_0^1 \{z^\alpha \overline{z}^\beta, z^\mu \overline{z}^\nu\} \circ \phi_0^t dt \\
& + \sum_{|\mu+\nu|=1} \langle \sigma_1\sigma_3 B_{\mu\nu}(\varrho(f)) | \int_0^1 (f\{z^\alpha \overline{z}^\beta, z^\mu \overline{z}^\nu\}) \circ \phi_0^t dt \rangle \big] \\
& + \widehat{\mathbf{R}}_1 + \int_0^1 (\langle V_j(\varrho(f)) f | f \rangle \widetilde{\chi}_j) \circ \phi^t dt,
\end{aligned}
\tag{12.25}
$$

where by Lemmas 11.3 and 11.4 and by $\widehat{\mathbf{R}}_1 = O(|z| + \|f\|_{\Sigma_{4k_0+4-4k_1}})^3$ s.t. $\widehat{\mathbf{R}}_1$ is C^{k_0-1} in $z \in \mathbb{C}^m$, $f \in \Sigma_{4k_0+4-4k_1}$ and $\varrho(f)$. In particular we have used estimates like

$$
\begin{aligned}
& |\langle \sigma_1\sigma_3 B_{\mu\nu}(\varrho(f)) | \mathcal{H} f \circ \phi^t - \mathcal{H} f \circ \phi_0^t \rangle| \le \|B_{\mu\nu}(\varrho(f))\|_{\Sigma_{4k_1}} \\
& \times \|\mathcal{H} f \circ \phi^t - \mathcal{H} f \circ \phi_0^t\|_{\Sigma_{-4k_1}} \le C \|f \circ \phi^t - f \circ \phi_0^t\|_{\Sigma_{2-4k_1}} \\
& \le C' (|z| + \|f\|_{\Sigma_{3-4k_1}})^2,
\end{aligned}
\tag{12.26}
$$

with the latter a consequence of Lemma 11.4. In (12.25) the last term is like in Lemma 10.3. It can be treated by Lemma 12.4. By our choice of $k_2(2)$ and $k_3(2)$, if we denote by $\widetilde{\mathbf{R}}_1$ the last line of (12.25), we conclude that $\widetilde{\mathbf{R}}_1$ can be absorbed in $\mathcal{R}_0^{(2)} + \mathcal{R}_1^{(2)} + \widehat{\mathcal{R}}_2^{(2)}$. We then obtain for $\ell = 1$

$$
\begin{aligned}
& H_2^{(\ell)} \circ \phi = H_2^{(\ell)} + \mathrm{i} \sum_{|\mu+\nu|=\ell+1} b_{\mu\nu}^{(\ell)}(\varrho(f)) \lambda(\varrho(f)) \cdot (\mu-\nu) z^\mu \overline{z}^\nu \\
& - \mathrm{i} \sum_{|\mu+\nu|=\ell} z^\mu \overline{z}^\nu \langle f | \sigma_1\sigma_3 (\mathcal{H} - \lambda \cdot (\mu-\nu)) B_{\mu\nu}^{(\ell)}(\varrho(f)) \rangle \\
& + \sum_{|\mu+\nu|=\ell+1} k_{\mu\nu}^{(\ell)}(\varrho(f), b(\varrho(f)), B(\varrho(f))) z^\mu \overline{z}^\nu \\
& + \sum_{|\mu+\nu|=\ell} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 K_{\mu\nu}^{(\ell)}(\varrho(f), b(\varrho(f)), B(\varrho(f))) | f \rangle + \mathbf{R}_\ell
\end{aligned}
\tag{12.27}
$$

where: $\ell = 1$; $\mathbf{R}_1$ is like $\widetilde{\mathbf{R}}_1$; $k_{\mu\nu}(0,b,B) = 0$ and $K_{\mu\nu}(0,b,B) = 0$ (follows by (9.8)); $k_{\mu\nu}(\varrho,b,B) \in \mathbb{C}$ and $K_{\mu\nu}(\varrho,b,B) \in \Sigma_{4k_1}$ are C^{k_0} in ϱ, in $b_{\mu\nu} \in \mathbb{C}$ and in

$B_{\mu\nu} \in \Sigma_{4k_1}$. Notice that to get C^{k_0} regularity it is crucial the use of ϕ_0^t and its properties stated under (11.13). The C^{k_0} regularity is key for the homological equation.

By Lemma 12.5 we have

$$\begin{aligned}\widetilde{\mathcal{R}^{(1)}} \circ \phi - \widetilde{\mathcal{R}^{(1)}} &= \sum_{|\mu+\nu|=2} \widetilde{k}_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f))) z^\mu \overline{z}^\nu \\ &+ \sum_{|\mu+\nu|=1} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 \widetilde{K}_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f))) | f\rangle + \widetilde{\mathbf{S}}_1,\end{aligned} \tag{12.28}$$

with: $\widetilde{\mathbf{S}}_1$ like $\mathcal{R}^{(2)}$; $\widetilde{k}_{\mu\nu}$ (resp. $\widetilde{K}_{\mu\nu}$) is like $k_{\mu\nu}$ (resp. $K_{\mu\nu}$).

By Lemma 11.5 we have

$$\psi(\varrho(f)) \circ \phi = \psi(\varrho(f)) \circ \phi_0 + \mathbf{T}_\ell$$

for $\ell = 1$ where $\mathbf{T}_1$ is like $\mathcal{R}^{(2)}$. By Lemma 8.7 applied to ϕ_0^t, exploiting the fact that $\psi(0) = 0$, we have that

$$\begin{aligned}&\psi(\varrho(f)) \circ \phi_0 - \psi(\varrho(f)) = \sum_{|\mu+\nu|=\ell+1} k_{\mu\nu}^{(\ell)\prime}(\varrho(f), b(\varrho(f)), B(\varrho(f))) z^\mu \overline{z}^\nu + \\ &\sum_{a=1}^{3}\sum_{i=0}^{1}\sum_{|\mu+\nu|=\ell} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 \partial_a^i K_{\mu\nu}^{(\ell)ia}(\varrho(f), b(\varrho(f)), B(\varrho(f))) | f\rangle + \mathbf{R}'_\ell\end{aligned} \tag{12.29}$$

with $\mathbf{R}'_\ell$ like $\mathbf{R}_\ell$.

By Lemma 12.6

$$\begin{aligned}&\langle B_2(z', \varrho(f')) | (f')^2\rangle = \widehat{\mathbf{S}}_1 \\ &+ \sum_{|\mu+\nu|=2} \upsilon_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f))) z^\mu \overline{z}^\nu \\ &+ \sum_{|\mu+\nu|=1} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 \Upsilon_{\mu\nu}(\varrho(f), b(\varrho(f)), B(\varrho(f))) | f\rangle,\end{aligned} \tag{12.30}$$

where $\upsilon_{\mu\nu}$ and $\Upsilon_{\mu\nu}$ have the same properties of $k_{\mu\nu}$, $K_{\mu\nu}$, $\widehat{\mathbf{S}}_1$ is like $\mathcal{R}^{(2)}$. Notice that the fact that $\upsilon_{\mu\nu}$ and $\Upsilon_{\mu\nu}$ are C^{k_0} is key here for the homological equation.

By Lemma 12.4, where we are using $k_0 \ll k_1 \ll M$, which is much more than needed, we have for $\ell = 1$

$$\begin{aligned}\widehat{\mathcal{R}}_2^{(\ell)} \circ \phi &= \widehat{\mathcal{R}}_2^{(\ell)} + \sum_{|\mu+\nu|=\ell+1} \kappa_{\mu\nu}^{(\ell)}(\varrho(f), b(\varrho(f)), B(\varrho(f))) z^\mu \overline{z}^\nu \\ &+ \sum_{|\mu+\nu|=\ell} z^\mu \overline{z}^\nu \langle \sigma_1\sigma_3 \mathcal{K}_{\mu\nu}^{(\ell)}(\varrho(f), b(\varrho(f)), B(\varrho(f))) | f\rangle + \widehat{\mathbf{S}}_\ell,\end{aligned} \tag{12.31}$$

where $\kappa_{\mu\nu}^{(1)}$, $\mathcal{K}_{\mu\nu}^{(1)}$ and $\widehat{\mathbf{S}}_1^{(1)}$ are like $k_{\mu\nu}$, $K_{\mu\nu}$ and $\mathcal{R}^{(2)}$. Consider now $\mathbf{K} := \widetilde{\mathcal{R}^{(1)}}$

and $\widetilde{\mathbf{K}}$ the polynomial of the form (10.9) with

$$\widetilde{\mathbf{k}}_{\mu\nu}(\varrho,b,B):=(k_{\mu\nu}+k^{(1)\prime}_{\mu\nu}+\widetilde{k}_{\mu\nu}+\upsilon_{\mu\nu}+\kappa^{(1)}_{\mu\nu})(\varrho,b,B)\,,$$
$$\widetilde{\mathbf{K}}_{\mu\nu}(\varrho,b,B):=(K_{\mu\nu}+\widetilde{K}_{\mu\nu}+\sum_{i=0}^{1}\sum_{a=1}^{3}\partial_a^i K^{(1)ia}_{\mu\nu}+\Upsilon_{\mu\nu}+\mathcal{K}^{(1)}_{\mu\nu})(\varrho,b,B).$$

Then $\mathbf{K}(\varrho)$ and $\widetilde{\mathbf{K}}(\varrho,b,B)$ are like in Lemmas 10.2-10.3. This means that we can choose χ in (11.1) so that

$$\begin{aligned}&\mathrm{i}\lambda(\varrho)\cdot(\mu-\nu)z^{\mu}\overline{z}^{\nu}b_{\mu\nu}+z^{\mu}\overline{z}^{\nu}\mathrm{i}\langle f|\sigma_1\sigma_3(\lambda(\varrho),\cdot(\mu-\nu)-\mathcal{H})B_{\mu\nu}\rangle\\&+\widetilde{\mathcal{R}^{(1)}}+\widetilde{\mathbf{K}}(\varrho(f),b(\varrho(f)),B(\varrho(f)))=Z_1(\varrho(f)),\end{aligned}$$

where $Z_1(\varrho)$ is in normal form and homogeneous of degree 2 in $(z,\overline{z},f)$ and with $Z_1(0)=0$.

For $d>2$, the terms $\langle B_d(z',\varrho(f'))|(f')^d\rangle$ can can be incorporated in $\mathcal{R}^{(2)}$ by Lemma 12.6. This is true also for the $d=5$ term and for $E_P(f')$. We set

$$\begin{aligned}\widetilde{\mathcal{R}}^{(2)}:&=\widehat{\mathcal{R}}^{(1)}_2+\mathbf{R}_1+\widehat{\mathbf{S}}_1+\widetilde{\mathbf{S}}_1+\mathbf{T}_1+\mathbf{R}_1'\\&+\widetilde{\mathcal{R}^{(2)}}\circ\phi-\langle B_2(z',\varrho(f'))|(f')^2\rangle-\widehat{\mathcal{R}}^{(1)}_2\circ\phi.\end{aligned}$$

All the terms in the rhs have the properties required to $\mathcal{R}^{(2)}$. Hence, if we also set $\mathcal{R}^{(2)}:=\widetilde{\mathcal{R}}^{(2)}$, we conclude the proof of case $\ell=2$ in Theorem 12.1.

12.3 Proof of Theorem 12.1: the step $\ell>2$

Case $\ell=2$ has been treated in Subsection 12.2. We proceed by induction to complete the proof of Theorem 12.1. From the argument below one can see that $H_2^{(\ell)}=H_2^{(2)}$ for all $\ell\geq 2$. Suppose that the statement of Theorem 12.1 holds for an $\ell\geq 2$.

Set $k_0=k_2(\ell)-\underline{k}$ and $k_1=k_3(\ell)-\underline{k}$ for a fixed and appropriately large $\underline{k}$. We will choose $2N\ll k_2(\ell+1)\ll k_3(\ell+1)\ll k_1-k_0$ with $k_2(\ell+1)$ as large as needed.

Since $H^{(\ell)}=H\circ\mathcal{T}_\ell$ is real valued (because H is real valued), $a^{(\ell)}_{\mu\nu}$ and $G^{(\ell)}_{\mu\nu}$ satisfy (4.12). We seek an appropriate polynomial χ as in (11.1) with $M_0=\ell$. For any such polynomial, we consider its Lie transform $\phi=\phi^1$. Proceeding like in the previous step of the proof, we obtain formula (12.27) with: $\mathbf{R}_\ell=O(|z|+\|f\|_{\Sigma_{-4k_1+2}})^{\ell+2}$, with $\mathbf{R}_\ell\in C^{k_0-1}$ in $z\in\mathbb{C}^m$, $f\in\Sigma_{4k_0-4k_1}$ and $\rho(f)$; $k^{(\ell)}_{\mu\nu}(0,b,B)=0$ resp. $K^{(\ell)}_{\mu\nu}(0,b,B)=0$; $\widehat{k}^{(\ell)}_{\mu\nu}(\varrho,b,B)\in\mathbb{C}$ and $\widehat{K}^{(\ell)}_{\mu\nu}(\varrho,b,B)\in\Sigma_{4k_1}$ are C^{k_0} in $\varrho\in\mathbb{R}^4$, $b_{\mu\nu}\in\mathbb{C}$ and $B_{\mu\nu}\in\Sigma_{4k_1}$. By our choice of $k_2(\ell+1)$ and $k_3(\ell+1)$, $\mathbf{R}_\ell$ can be absorbed in $\mathcal{R}^{(\ell+1)}_0+\mathcal{R}^{(\ell+1)}_1+\widehat{\mathcal{R}}^{(\ell+1)}_2$.

By Lemma 12.6 we have

$$
\begin{aligned}
&\langle F_2^{(\ell)}(z',f',f'(\cdot),\varrho(f'))|(f')^2\rangle = \langle F_2^{(\ell)}(z,f,f(\cdot),\varrho(f))|f^2\rangle \\
&+\widehat{\mathbf{S}}_\ell + \sum_{|\mu+\nu|=\ell+1} \upsilon^{(\ell)}_{\mu\nu}(\varrho(f),b(\varrho(f)),B(\varrho(f)))z^\mu\overline{z}^\nu \\
&+\sum_{|\mu+\nu|=\ell} z^\mu\overline{z}^\nu\langle\sigma_1\sigma_3\Upsilon^{(\ell)}_{\mu\nu}(\varrho(f),b(\varrho(f)),B(\varrho(f)))|f\rangle,
\end{aligned}
\tag{12.32}
$$

where: $\upsilon^{(\ell)}_{\mu\nu}$ (resp. $\Upsilon^{(\ell)}_{\mu\nu}$) is like $k^{(\ell)}_{\mu\nu}$ (resp. $K^{(\ell)}_{\mu\nu}$); $\widehat{\mathbf{S}}_\ell$ is like $\mathcal{R}^{(\ell+1)}$. Proceeding like in (12.32), by Lemma 12.4 we get

$$
\begin{aligned}
&\widehat{\mathcal{R}}^{(\ell)}_2\circ\phi-\widehat{\mathcal{R}}^{(\ell)}_2 = \sum_{|\mu+\nu|=\ell+1}\kappa^{(\ell)}_{\mu\nu}(\varrho(f),b(\varrho(f)),B(\varrho(f)))z^\mu\overline{z}^\nu \\
&+\sum_{|\mu+\nu|=\ell} z^\mu\overline{z}^\nu\langle\sigma_1\sigma_3\mathcal{K}^{(\ell)}_{\mu\nu}(\varrho(f),b(\varrho(f)),B(\varrho(f)))|f\rangle+\widehat{\mathbf{S}}_\ell,
\end{aligned}
\tag{12.33}
$$

with $\widetilde{\mathbf{S}}_\ell$ like $\mathcal{R}^{(\ell+1)}$, $\widetilde{k}^{(\ell)}_{\mu\nu}$ resp. $\widetilde{K}^{(\ell)}_{\mu\nu}$ with the properties of $k^{(\ell)}_{\mu\nu}$ resp. $K^{(\ell)}_{\mu\nu}$. Proceeding as for the $\ell=2$ case, we have that $\psi(\varrho(f))\circ\phi-\psi(\varrho(f))$ is like the right hand side of (12.29).
Set $\mathbf{K}^{(\ell)}(\varrho(f)) := \mathcal{R}^{(\ell)}_0+\mathcal{R}^{(\ell)}_1$. Consider the polynomial $\widetilde{\mathbf{K}}$ of the form (10.9) with coefficients

$$
\begin{aligned}
&\widetilde{\mathbf{k}}^{(\ell)}_{\mu\nu}(\varrho,b,B) := (k^{(\ell)}_{\mu\nu}+k^{(\ell)\prime}_{\mu\nu}+\widetilde{k}^{(\ell)}_{\mu\nu}+\upsilon^{(\ell)}_{\mu\nu})(\varrho,b,B)\,,\\
&\widetilde{\mathbf{K}}^{(\ell)}_{\mu\nu}(\varrho,b,B) := (K^{(\ell)}_{\mu\nu}+\sum_{i=0}^{1}\sum_{a=1}^{3}\partial^i_a K^{(\ell)ia}_{\mu\nu}+\mathcal{K}^{(\ell)}_{\mu\nu}+\Upsilon^{(\ell)}_{\mu\nu})(\varrho,b,B).
\end{aligned}
$$

Then $\mathbf{K}^{(\ell)}(\varrho)$ and $\widetilde{\mathbf{K}}^{(\ell)}(\varrho,b,B)$ are like in Lemma 10.2. This means that we can choose χ in (11.1) so that

$$
\begin{aligned}
&\mathrm{i}\lambda(\varrho)\cdot(\mu-\nu)z^\mu\overline{z}^\nu b_{\mu\nu}+z^\mu\overline{z}^\nu\mathrm{i}\langle f|\sigma_1\sigma_3(\lambda(\varrho),\cdot(\mu-\nu)-\mathcal{H})B_{\mu\nu}\rangle\\
&+\mathbf{K}^{(\ell)}(\varrho)+\widetilde{\mathbf{K}}^{(\ell)}(\varrho,b,B)=Z_\ell(\varrho),
\end{aligned}
$$

where $Z_{\ell+1}(\varrho)$ is in normal form and homogeneous of degree $\ell+1$ in $(z,\overline{z},f)$. Set $Z^{(\ell+1)} := Z^{(\ell)}+Z_{\ell+1}$ and

$$
H^{(\ell+1)} := H^{(\ell)}\circ\phi = H^{(\ell)}_2+Z^{(\ell+1)}+\widetilde{\mathcal{R}}^{(\ell+1)},
\tag{12.34}
$$

where

$$
\begin{aligned}
\widetilde{\mathcal{R}}^{(\ell+1)} :&= Z^{(\ell+1)}\circ\phi-Z^{(\ell+1)}\\
&+\sum_{d=3}^{6}\mathcal{R}^{(\ell)}_d\circ\phi+\widetilde{\mathcal{R}}^{(\ell)}_2+\mathbf{R}_\ell+\widetilde{\mathbf{S}}_\ell+\widehat{\mathbf{S}}_\ell+\mathbf{R}'_\ell.
\end{aligned}
$$

The terms in the last line are of the form requested for terms of $\mathcal{R}^{(\ell+1)}$. By Lemma 12.5 also $Z^{(\ell+1)}\circ\phi-Z^{(\ell+1)}$ is of the same type. Then we set $\mathcal{R}^{(\ell+1)} := \widetilde{\mathcal{R}}^{(\ell+1)}$ and the proof is finished.

$\square$

13 Dispersion

We apply Theorem 12.1 for $\ell = 2N+1$ (recall $N = N_1$ where $N_j\lambda_j < \omega_0 < (N_j+1)\lambda_j$). In the rest of the paper we work with the hamiltonian $H^{(2N+1)}$. We will drop the upper index. So we will set $H = H^{(2N+1)}$, $H_2 = H_2^{(2N+1)}$, $\lambda_j = \lambda_j^{(2N+1)}$, $\lambda = \lambda^{(2N+1)}$, $Z_i = Z_i^{(2N+1)}$ for $i = 0, 1$ and $\mathcal{R} = \mathcal{R}^{(2N+1)}$. In particular we will denote by $G_{\mu\nu}$ the coefficients $G_{\mu\nu}^{(2N+1)}$ of $Z_1^{(2N+1)}$. We will show:

Theorem 13.1. *Consider the constant $0 < \epsilon < \varepsilon$ of Theorem 9.1. There is a fixed $C > 0$ such that for ε sufficiently small and for any $\epsilon \in (0, \varepsilon)$ we have*

$$\|f\|_{L_t^p([0,\infty),W_x^{1,q})} \leq C\epsilon \text{ for all admissible pairs } (p,q), \tag{13.1}$$

$$\|z^\mu\|_{L_t^2([0,\infty))} \leq C\epsilon \text{ for all multi indexes } \mu \text{ with } \lambda\cdot\mu > \omega_0, \tag{13.2}$$

$$\|z_j\|_{W_t^{1,\infty}([0,\infty))} \leq C\epsilon \text{ for all } j \in \{1,\dots,m\}\ . \tag{13.3}$$

Notice that by the time reversibility of the NLS, the above estimates imply the ones with $\mathbb{R}$ replacing $[0,\infty)$, doubling constants in (13.1)–(13.2).

(13.3) is a consequence of the already known orbital stability, so we do not need to prove it. To obtain Theorem 13.1 it is enough to show that there are fixed constants C_1, C_2 (large) and ε (small) such that if for $\epsilon \in (0,\varepsilon)$ (where ϵ and ε are those of Theorem 9.1)

$$\|f\|_{L_t^p([0,T],W_x^{1,q})} \leq C_1\epsilon \text{ for all admissible pairs } (p,q), \tag{13.4}$$

$$\|z^\mu\|_{L_t^2([0,T])} \leq C_2\epsilon \text{ for all multi indexes } \mu \text{ with } \omega\cdot\mu > \omega_0, \tag{13.5}$$

then in fact (13.4) and (13.5) hold but with C_1, C_2 replaced by $C_1/2$, $C_2/2$. In fact we conclude that these estimates hold for all T and so (13.1)–(13.2) hold. The proof consists in three main steps.

(i) Estimate f in terms of z.

(ii) Substitute the variable f with a new "smaller" variable g and find smoothing estimates for g.

(iii) Reduce the system for z to a closed system involving only the z variables, by insulating the part of f which interacts with z, and by decoupling the rest (this reminder is g). Then clarify the nonlinear Fermi golden rule.

13.1 Proof of Theorem 13.1: step (i)

Step (i) is encapsulated by the following proposition:

Proposition 13.2. *Assume* (13.4)–(13.5). *Then there exist constants $C = C(C_1, C_2)$ and $K_1 = K_1(C_1)$, such that, if $C(C_1,C_2)\epsilon$ is sufficiently small, then we have*

$$\|f\|_{L_t^p([0,T],W_x^{1,q})} \leq K_1\epsilon \text{ for all admissible pairs } (p,q)\ . \tag{13.6}$$

Proof. Consider Z_1 of the form (10.4). Set:

$$G^0_{\mu\nu} = G_{\mu\nu}(\varrho(0)); \quad \lambda^0_j = \lambda_j(\omega_0). \tag{13.7}$$

Then we have (with finite sums)

$$\begin{aligned} &\mathrm{i}\dot{f} - \mathcal{H}f - (\partial_{Q(f)}H)P_c(\omega_0)\sigma_3 f - \mathrm{i}(\partial_{\Pi_a(f)}H)P_c(\omega_0)\partial_{x_a} f \\ &= \sum_{\substack{|\lambda^0\cdot(\nu-\mu)|>\omega_0,\\ |\mu+\nu|\le 2N_1+1}} z^\mu \overline{z}^\nu G^0_{\mu\nu} \\ &+ \sum_{\substack{|\lambda^0\cdot(\nu-\mu)|>m-\omega_0,\\ |\mu+\nu|\le 2N_1+1}} z^\mu \overline{z}^\nu (G_{\mu\nu} - G^0_{\mu\nu}) + \sigma_3\sigma_1 \widehat{\nabla}_f \mathcal{R}, \end{aligned} \tag{13.8}$$

with $\widehat{\nabla}_f \mathcal{R}(z,f,\rho)$ the gradient in f, with no differentiation in $\varrho(f)$. In order to obtain bounds on f, we need bounds on the right hand term of the equation especially the last two terms. They are provided by the following lemma.

Lemma 13.3. *Assume* (13.4)–(13.5). *Then there is a constant* $C(C_1,C_2)$ *independent of* ϵ *such that the following is true: we have* $\sigma_3\sigma_1\widehat{\nabla}_f\mathcal{R} = R_1 + R_2$ *with*

$$\|R_1\|_{L^1_t([0,T],H^1_x)} + \|R_2\|_{L^2_t([0,T],W^{1,\frac{6}{5}}_x)} \le C(C_1,C_2)\epsilon^2. \tag{13.9}$$

Proof. The proof is standard, a combination of [BC] and [CM]. □

Lemma 13.4. *Consider* $\mathrm{i}\dot{\psi} - \mathcal{H}\psi - \varphi(t)\sigma_3 P_c\psi - \mathrm{i}A_a(t)P_c\partial_{x_a}\psi = F$ *where:* $P_c = P_c(\omega_0)$*,* $\psi = P_c\psi$*,* φ *and each* A_a *are real valued. Then there exist* $c_0 > 0$ *and* $C > 0$ *such that if* $\|(\varphi, A)\|_{L^\infty_t[0,T]} < c_0$ *then for* (p,q) *as in Theorem 13.1 we have*

$$\|\psi\|_{L^p_t([0,T],W^{1,q})} \le C\|\psi(0)\|_{H^1} + C\|F\|_{L^1_t([0,T],H^1_x)+L^2_t([0,T],W^{1,\frac{6}{5}}_x)} \tag{13.10}$$

Proof. This result is due to Beceanu, see for example Theorem 3.8 [Be]. □

Continuation of the proof of Proposition 13.2. By (13.8) we can apply to f Lemma 13.4 by taking $\varphi(t) = \partial_{Q(f)}H$, $A_a(t) = \partial_{\Pi_a(f)}H$ and $F = \mathrm{rhs}(13.8) - \varphi[\sigma_3, P_d]f$. Then, for fixed constants

$$\begin{aligned} &\|f\|_{L^p_t([0,T],W^{1,q})} \le C_1\|f(0)\|_{H^1} + C_1\|F\|_{L^1_t([0,T],H^1_x)+L^2_t([0,T],W^{1,\frac{6}{5}}_x)} \\ &\le C_1\|f(0)\|_{H^1} + C\sum_{\lambda\cdot\mu>m-\omega_0} \|z^\mu\|^2_{L^2_t(0,T)} \\ &+ C\|R_1\|_{L^1_t([0,T],H^1_x)} + \|R_2\|_{L^2_t([0,T],W^{1,\frac{6}{5}}_x)} + C\epsilon\|f\|_{L^2_t([0,T],L^6_x)}. \end{aligned} \tag{13.11}$$

For ϵ small this yields Proposition 13.2 by Lemma 13.4 and by (13.5). □

Lemma 13.5. *Assume the conclusions of Theorem 13.1. Then there exists a fixed* $C > 0$ *and* $f_+ \in H^1$ *with* $\|f_+\|_{H^1} < C\epsilon$ *such that we have*

$$\lim_{t\to+\infty} \left\| \tau_{X(t)} e^{\mathrm{i}\chi(t)\sigma_3} f(t) - e^{\mathrm{i}t\Delta\sigma_3} f_+ \right\|_{H^1} = 0 \tag{13.12}$$

for $\chi(t) := t\omega_0 + \int_0^t \partial_{Q(f)} H(t')dt'$ *and* $X(t) := \int_0^t \partial_{\Pi(f)} H((t')dt'$.

Proof. For $\psi(t) = f(t)$, $F = \text{rhs}(13.8) - \varphi(t)[\sigma_3, P_d]f$, $\varphi(t) = \partial_{Q(f)} H$, $\mathcal{A}_a(t) = \partial_{\Pi_a(f)} H$, $\mathcal{U}(t) = e^{\int_0^t (\mathrm{i}\sigma_3\varphi(\tau) + \mathcal{A}(\tau)\cdot\nabla)d\tau}$ and for $t_1 < t_2$, we have

$$\begin{aligned}
&\|e^{\mathrm{i}\mathcal{H}_0 t_2}\mathcal{U}(t_2)f(t_2) - e^{\mathrm{i}\mathcal{H}_0 t_1}\mathcal{U}(t_1)f(t_1)\|_{H^1} \le \\
&\| \int_{t_1}^{t_2} e^{\mathrm{i}\mathcal{H}_0 t'}\mathcal{U}(t') \left[F(t') + Vf(t') - \varphi(t')\sigma_3 P_d f(t') - \mathrm{i}\mathcal{A}_a P_d \partial_a f(t')\right] dt' \|_{H^1} \\
&\le C(\sum_{|\lambda^0\cdot\mu| > m-\omega_0} \|z^\mu\|_{L^2(t_1,t_2)} + \|R_1\|_{L^1_t([t_1,t_2],H^1_x)} \\
&+ \|R_2\|_{L^2_t([t_1,t_2],W_x^{1,\frac{6}{5}})} + \|f\|_{L^2_t([t_1,t_2],W_x^{1,6})}).
\end{aligned}$$

Since the rhs has limit 0 as $t_1 \to +\infty$, there exists $f_+ \in H^1$ such that

$$\lim_{t\to+\infty} \left\| \mathcal{U}(t)f(t) - e^{-\mathrm{i}\mathcal{H}_0 t} f_+ \right\|_{H^1} = 0.$$

This yields Lemma 13.5. ☐

Lemma 13.6. *Assume the conclusions of Theorem 13.1 and the notation of Theorem 9.1. Then the conclusions of Theorem 9.1 hold with the* f_+ *of* (13.12) *and with*

$$\widehat{\vartheta} = \chi + \mathrm{i} \sum_{\ell=1}^{2N+1} \gamma_\ell \,, \quad \widehat{D} = X - \sum_{\ell=1}^{2N+1} \mathbf{A}_\ell, \tag{13.13}$$

with $\mathbf{A}_\ell$ *and* γ_ℓ *the terms in* (11.6) *corresponding to the lie transforms* ϕ_ℓ *of Theorem 12.1.*

Proof. This follows immediately from (11.6). Indeed, schematically

$$e^{\mathrm{i}\chi - \sum_{\ell=1}^{2N+1}\gamma_\ell} \tau_{X - \sum_{\ell=1}^{2N+1}\mathbf{A}_\ell} f_{9.1} \approx e^{\mathrm{i}\chi} \tau_X f_{13.1} \approx e^{\mathrm{i}t\Delta\sigma_3} f_+,$$

where $f_{9.1}$ (resp. $f_{13.1}$) is the coordinate in Theorem 9.1 (resp. Theorem13.1). ☐

Lemma 13.7. *Denote by* (ω, v, z', f') *the coordinates* (2.13) *of the solution in the initial system of coordinates (we omit* ϑ, D*). Then* $\lim_{t\to\infty} z'(t) = 0$*, there are functions* $\theta \in C^1(\mathbb{R}, \mathbb{R})$ *and* $y \in C^1(\mathbb{R}, \mathbb{R}^3)$ *s.t.*

$$\lim_{t\to+\infty} \left\| \tau_{y(t)} e^{\mathrm{i}\sigma_3\left(v(t)\cdot\frac{x}{2} + \theta(t)\right)} f'(t) - e^{\mathrm{i}t\Delta\sigma_3} f_+ \right\|_{H^1} = 0, \tag{13.14}$$

with the f_+ *of Lemma 13.5, and* $\lim_{t\nearrow\infty} \omega(t) = \omega_+$ *and* $\lim_{t\nearrow\infty} v(t) = v_+$.

Proof. If we denote by (z', f') the initial coordinates and by $(z_{9.1}, f_{9.1})$ the coordinates in (9.2) considered in Theorem 9.1, we have $z' = z + O(|z_{9.1}| + \|f_{9.1}\|_{L^{2,-2}_x})$. So the asymptotic behavior of z' and of $z_{9.1}$ is the same. By (8.15) we get

$$\tau_{\widehat{D}} e^{\mathrm{i}\widehat{\vartheta}\sigma_3} f_{9.1} = \tau_{\widehat{D}-\mathbf{A}} e^{\sigma_3(\frac{\mathrm{i}}{2}v\cdot x - \widetilde{\gamma} + \mathrm{i}\widehat{\vartheta})} f' - \tau_{\widehat{D}-\mathbf{A}} e^{\sigma_3(\frac{\mathrm{i}}{2}v\cdot x - \widetilde{\gamma} + \mathrm{i}\widehat{\vartheta})} \mathcal{G}. \tag{13.15}$$

By (8.11) the second term on the rhs converges to 0 in H^1 as $t \nearrow \infty$. Hence, for $\theta := \widehat{\vartheta} + \mathrm{i}\widetilde{\gamma}$ and for $y := \widehat{D} - \mathbf{A}$, we obtain (13.14).

We have $q(\omega(t)) = q(\omega_0) - \frac{\|f'(t)\|_2^2}{2} + O(|z'(t)| + \|f'(t)\|_{L^{2,-2}_x})$ by $q(\omega_0) = q(\omega) + Q(R)$. Then (13.14) and $|z'(t)| + \|f'(t)\|_{L^{2,-2}_x} \to 0$ imply

$$\lim_{t\to+\infty} q(\omega(t)) = q(\omega_0) - \lim_{t\to+\infty} \frac{\|e^{\mathrm{i}t\sigma_3\Delta} f_+\|_2^2}{2} = q(\omega_0) - \frac{\|f_+\|_2^2}{2} = q(\omega_+),$$

where ω_+ is the unique element near ω_0 for which the last equality holds. So $\lim_{t\to+\infty}\omega(t) = \omega_+$. By $v = 2(\Pi(U_0) - \Pi(R))Q^{-1}(U_0)$ we obtain

$$v = 2(\Pi(U_0) - \Pi(f'))Q^{-1}(U_0) + O(|z'(t)| + \|f'(t)\|_{L^{2,-2}_x})$$

which implies $\lim_{t\nearrow\infty} v(t) = 2(\Pi(U_0) - \Pi(f_+))Q^{-1}(U_0) =: v_+$. $\square$

Lemma 13.8. *For (θ, y) the functions of (13.14) and (ϑ, D) the coordinates of (2.11), there are $\vartheta_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^3$ and $o(1) \to 0$ as $t \to +\infty$ s.t.*

$$\theta(t) = \vartheta(t) + \vartheta_0 + o(1), \quad y(t) = D(t) + y_0. \tag{13.16}$$

Proof. Consider the representation $U = \tau_D e^{\mathrm{i}\sigma_3(\frac{v\cdot x}{2}+\vartheta)}(\Phi_\omega + R)$ of the solution of $\mathrm{i}\dot{U} = \sigma_3\sigma_1\nabla E(U)$. We have the identity

$$\begin{aligned}\mathrm{i}\dot{U} &= -\sigma_3(\dot{\vartheta} - \frac{v\cdot\dot{D}}{2})\tau_D e^{\mathrm{i}\sigma_3\Theta}(\Phi_\omega + R) - \mathrm{i}\dot{D}\cdot\tau_D e^{\mathrm{i}\sigma_3\Theta}\nabla_x(\Phi_\omega + R)\\ &- \frac{\dot{v}}{2}\cdot\tau_D e^{\mathrm{i}\sigma_3\Theta}\sigma_3 x(\Phi_\omega + R) + \mathrm{i}\dot{\omega}\tau_D e^{\mathrm{i}\sigma_3\Theta}\partial_\omega\Phi_\omega + \mathrm{i}\tau_D e^{\mathrm{i}\sigma_3\Theta}\dot{R}.\end{aligned}$$

By Lemma 4.2 we have $\nabla E(U) =$

$$= \tau_D e^{-\mathrm{i}\sigma_3(\frac{v\cdot x}{2}+\vartheta)}\left(\nabla E(\Phi_\omega + R) - v_a\nabla\Pi_a(\Phi_\omega + R) + \frac{v^2}{4}\nabla Q(\Phi_\omega + R)\right).$$

Then, using also (2.4), $\mathrm{i}\dot{U} = \sigma_3\sigma_1\nabla E(U)$ can be expanded for $\varpi = \omega_0$ into

$$\begin{aligned}&-\sigma_3(\dot{\vartheta} - \frac{v\cdot\dot{D}}{2} + \frac{v^2}{4} - \varpi)(\Phi_\omega + R) - \mathrm{i}(\dot{D} - v)\cdot\nabla_x(\Phi_\omega + R) + \mathrm{i}\dot{R}\\ &- \frac{\dot{v}\cdot x}{2}\sigma_3(\Phi_\omega + R) + \mathrm{i}\dot{\omega}\partial_\omega\Phi_\omega = \sigma_3\sigma_1\left(\nabla E(\Phi_\omega + R) + \varpi Q(\Phi_\omega + R)\right).\end{aligned} \tag{13.17}$$

Using the first system of coordinates (2.14), and denoting the f–coordinate by f', we have for $\|G_1\|_{L^\infty_t([0,+\infty),L^1_x)} \le C\epsilon^2$

$$
\begin{aligned}
&-\sigma_3(\dot\vartheta - \frac{v\cdot\dot D}{2} + \frac{v^2}{4} - \omega_0)f' - \mathrm{i}(\dot D - v)\cdot\nabla_x f' - \frac{\dot v\cdot x}{2}\sigma_3 f' + \mathrm{i}\dot f' \\
&= \sigma_3(-\Delta+\omega_0)f' + G_1 .
\end{aligned}
\tag{13.18}
$$

Now we substitute in (13.18) the variables of the last coordinate system. In particular f' and f are related by a formula like (8.15):

$$
\begin{aligned}
&f'(x) = e^{\mathrm{i}\sigma_3(-\frac12 v\cdot(x-\mathbb{A})-\mathrm{i}\widehat\gamma)}f(x-\mathbb{A}) + \mathcal{G}(x)\ , \\
&\widehat\gamma = \widetilde\gamma + \sum_{\ell=1}^{2N+1}\gamma_\ell\,, \quad \mathbb{A} = \mathbf{A} + \sum_{\ell=1}^{2N+1}\mathbf{A}_\ell\,,
\end{aligned}
\tag{13.19}
$$

with the $(\widetilde\gamma, \mathbf{A})$ of (13.15) and the $(\gamma_\ell, \mathbf{A}_\ell)$ of Lemma 13.6. Substituting (13.19) in (13.18) we get after various cancelations, for $\|G_2\|_{L^\infty_t([0,+\infty),L^1_x\cap H^1_x)} \le C\epsilon$,

$$
-\sigma_3(\dot\vartheta-\omega_0)f - \mathrm{i}(\dot D + \dot{\mathbb{A}})\cdot\nabla f + \mathrm{i}\dot f = \sigma_3(-\Delta+\omega_0)f + G_2. \tag{13.20}
$$

We claim that equation (13.20) is equivalent to equation (13.8). Indeed, taking their difference we have

$$
a_0(t)\sigma_3 f + \mathrm{i}a_j(t)\partial_{x_j}f = \mathbf{G}
$$

with $\mathbf{G}$ (resp . a_j) a continuous functional with values $L^\infty(\mathbb{R}, L^1(\mathbb{R}^3)\cap H^1_x)$ (resp . $L^\infty(\mathbb{R})$) bounded in the space of solutions we are considering. Then $a_0(t)\int f(t,x)dx = \sigma_3\int \mathbf{G}dx$. If $a_0(t_0)\neq 0$ for a given solution, we can find solutions for which $f_n(t,\cdot)\in\mathcal{S}(\mathbb{R}^3)$, $f_n(t_0,\cdot)\to f(t_0,\cdot)$ in $H^1(\mathbb{R}^3)$, $\|f_n(t_0)\|_{L^1(\mathbb{R}^3)}\nearrow\infty$, $\mathbf{G}_n(t_0)\to\mathbf{G}(t_0)$ and $a_{0n}(t_0)\to a_0(t_0)$. This yields a contradiction. So $a_0(t)\equiv 0$. By similar reasons $a_j(t)\equiv 0$. This implies $\mathbf{G}\equiv 0$. Equivalence of (13.20) and (13.8) yields

$$
\dot\vartheta + \frac12\frac{d}{dt}(v\cdot\mathbb{A}) - \mathrm{i}\dot{\widehat\gamma} = \dot\chi\,, \quad \dot D_a + \dot{\mathbb{A}}_a = \partial_{\Pi_a(f)}H\ , \tag{13.21}
$$

with the first one a consequence of $\dot\vartheta - \omega_0 = \partial_{Q(f)}H$. Using the notation of Lemmas 13.5–13.7, (13.21) yields what follows:

$$
\begin{aligned}
&\dot\theta = \dot\vartheta + \frac12\frac{d}{dt}(v\cdot\mathbb{A})\,; \\
&\dot D + \dot{\mathbb{A}} = \dot X = \dot{\widehat D} + \sum_{\ell=1}^{2N+1}\dot{\mathbf{A}}_\ell = \dot y + \dot{\mathbb{A}} \text{ and so } \dot y = \dot D.
\end{aligned}
\tag{13.22}
$$

Notice that by Lemma 8.4 and inequality (11.8) there exists $\lim_{t\infty}\mathbb{A}(t)$. By Lemma 13.7 we have $\lim_{t\nearrow\infty} v(t) = v_+$. This proves the existence of ϑ_0 in (13.16), which is then proved. $\square$

Lemma 13.9. *The functions (ϑ, D) in Lemma 13.8 satisfy $\dot{D} = v + o(1)$ and $\dot{\vartheta} = \omega + \frac{v^2}{4} + o(1)$, with $\lim_{t\to\infty} o(1) = 0$.*

Proof. Consider (13.17) with $\varpi = \omega$. Applying to it the linear operator $|x_a \Phi_\omega\rangle$ and using the (z, f) of the initial coordinate system (2.14) we get $|\dot{D}_a - v_a| \leq C(|z| + \|f\|_{L^{2,-S}})$ for arbitrary S and fixed C. So the rhs is o(1) and we conclude $\dot{D} = v + o(1)$. Applying $|\sigma_3 \partial_\omega \Phi_\omega\rangle$ to (13.17) we get similarly $|\dot{\vartheta} - \frac{v\cdot\dot{D}}{2} + \frac{v^2}{4} - \omega| \leq C(|z| + \|f\|_{L^{2,-S}})$. Using $\dot{D} = v + o(1)$ we conclude $\dot{\vartheta} = \omega + \frac{v^2}{4} + o(1)$. □

13.2 Steps (ii) and (iii): the Fermi golden rule

Step (ii) in the proof of Theorem 13.1 consists in introducing the variable

$$g = f + \sum_{|\lambda^0\cdot(\mu-\nu)|>\omega_0} z^\mu \overline{z}^\nu R^+_{\mathcal{H}}(\lambda^0 \cdot (\mu - \nu)) G^0_{\mu\nu}. \tag{13.23}$$

Substituting the new variable g in (13.8), the first line on the rhs of (13.8) cancels out. The following result has been proved in a variety of places in the absence of translation, see for example [BuC] . Thanks to Sect. 3.3 [Be], essentially the same proof holds here. We skip the proof.

Lemma 13.10. *For ϵ in Theorem 9.1 sufficiently small, for $C_0 = C_0(\mathcal{H})$ a fixed constant, we have $\|g\|_{L^2_t L^{2,-S}_x} \leq C_0\epsilon + O(\epsilon^2)$ for a fixed $S > 1$.*

We have arrived at step (iii) of the proof of Theorem 13.1: the Fermi Golden Rule.

Proposition 13.11. *There is a new set of variables $\zeta = z + O(z^2)$ such that for a fixed C we have*

$$\|\zeta - z\|_{L^2_t} \leq CC_2\epsilon^2 , \quad \|\zeta - z\|_{L^\infty_t} \leq C\epsilon^3 \tag{13.24}$$

and we have

$$\begin{aligned} \partial_t \sum_{j=1}^m \lambda^0_j |\zeta_j|^2 &= 2\sum_{j=1}^m \lambda^0_j \operatorname{Im}\left(\mathcal{D}_j \overline{\zeta}_j\right) - \\ &- 2 \sum_{\substack{\lambda^0\cdot\alpha=\lambda^0\cdot\nu>\omega_0 \\ \lambda\cdot\alpha-\lambda_k<\omega_0\,\forall k \;\; s.t.\;\; \alpha_k\neq 0 \\ \lambda\cdot\nu-\lambda_k<\omega_0\,\forall k \;\; s.t.\;\; \nu_k\neq 0}} \lambda^0 \cdot \nu \operatorname{Im}\left(\zeta^\alpha \overline{\zeta}^\nu \langle R^+_{\alpha 0} G^0_{\alpha 0} | \sigma_1\sigma_3 G^0_{0\nu}\rangle\right) \end{aligned} \tag{13.25}$$

where $\sum_j \|\mathcal{D}_j \overline{\zeta}_j\|_{L^1[0,T]} \leq (1 + C_2)c_0\epsilon^2$ for a fixed constant c_0.

Proof. See [Cu1] and [Cu4]. □

For the sum in the second line of (13.25) we get finite sums

$$
\begin{aligned}
&2\sum_{\Lambda>\omega_0}\Lambda\,\mathrm{Im}\left\langle R^+_{\mathcal{H}}(\Lambda)\sum_{\lambda^0\cdot\alpha=\Lambda}\zeta^\alpha G^0_{\alpha 0}|\sigma_1\sigma_3\sum_{\lambda^0\cdot\nu=\Lambda}\overline{\zeta}^\nu G^0_{0\nu}\right\rangle=\\
&2\sum_{\Lambda>\omega_0}\Lambda\,\mathrm{Im}\left\langle R^+_{\mathcal{H}}(\Lambda)\sum_{\lambda^0\cdot\alpha=\Lambda}\zeta^\alpha G^0_{\alpha 0}|\sigma_3\overline{\sum_{\lambda^0\cdot\alpha=\Lambda}\zeta^\alpha G^0_{\alpha 0}}\right\rangle,
\end{aligned}
\tag{13.26}
$$

where we have used $G^0_{\mu\nu}=-\sigma_1\overline{G^0}_{\nu\mu}$. Notice that the existence of $R^+_{\mathcal{H}}(\Lambda)$ for $\Lambda\in\sigma_e(\mathcal{H})$ is proved in [CPV].

We have:

Lemma 13.12 (Semipositivity)**.** *We have rhs*(13.26)≥ 0.

Proof. See [Cu1]. □

Now we will assume the following hypothesis.

(H10) We assume that for some fixed constants for any vector $\zeta\in\mathbb{C}^n$ we have:

$$
\sum_{\substack{(\alpha,\nu)\text{ as}\\ \text{in (13.25)}}}\lambda^0\cdot\nu\,\mathrm{Im}\left(\zeta^\alpha\overline{\zeta}^\nu\langle R^+_{\alpha 0}G^0_{\alpha 0}|\sigma_1\sigma_3G^0_{0\nu}\rangle\right)\approx\sum_{\substack{\lambda^0\cdot\alpha>\omega_0\\ \lambda^0\cdot\alpha-\lambda^0_k<\omega_0\\ \forall\,k\text{ s.t. }\alpha_k\neq 0}}|\zeta^\alpha|^2.
$$

By (H10) we have

$$
2\sum_j\lambda^0_j\,\mathrm{Im}\left(\mathcal{D}_j\overline{\zeta}_j\right)\gtrsim\partial_t\sum_j\lambda^0_j|\zeta_j|^2+\sum_{\alpha\text{ as in (H10)}}|\zeta^\alpha|^2.
$$

Then, for $t\in[0,T]$, by the last line in Proposition 13.11 we have

$$
\textstyle\sum_j\lambda^0_j|\zeta_j(t)|^2+\sum_{\alpha\text{ as in (H10)}}\|\zeta^\alpha\|^2_{L^2(0,t)}\lesssim\epsilon^2+C_2\epsilon^2.
$$

By (13.24) this implies $\|z^\alpha\|^2_{L^2(0,t)}\lesssim\epsilon^2+C_2\epsilon^2$ for all the above multi indexes. So, from $\|z^\alpha\|^2_{L^2(0,t)}\leq C_2^2\epsilon^2$ we conclude $\|z^\alpha\|^2_{L^2(0,t)}\leq\kappa C_2\epsilon^2$ for a fixed κ, which is an improvement if C_2 is sufficiently large. So if we take $C_1>2K_1(C_2)$ and C_1- C_2 sufficiently large, in particular so that $\sqrt{\kappa C_2}<C_2/2$, we conclude as desired that (13.4) and (13.5) imply the same estimate but with C_1, C_2 replaced by $C_1/2$, $C_2/2$. This yields Theorem 13.1.

Remark 13.13. Suppose for simplicity that $\lambda_1(\omega)=...=\lambda_m(\omega)=:\lambda(\omega)$ and let us see the meaning of (H10). We have $G^0_{\alpha 0}\in\mathcal{S}(\mathbb{R}^3,\mathbb{C}^2)$ for all α. For $W(\omega)=\lim_{t\to+\infty}e^{-\mathrm{i}t\mathcal{H}_\omega}e^{\mathrm{i}t\sigma_3(-\Delta+\omega)}$, there exist $F_\alpha\in W^{k,p}(\mathbb{R}^3,\mathbb{C}^2)$ for all $k\in\mathbb{R}$ and $p\geq 1$ with $G^0_{\alpha 0}=W(\omega_0)F_\alpha$, [Cu6]. Let ${}^tF_\alpha=(F^{(1)}_\alpha,F^{(2)}_\alpha)$. Then the left hand side of (H10) can be expressed as

$$
\int_{|\xi|=\sqrt{(N+1)\lambda_0-\omega_0}}\Big|\sum_{|\alpha|=N+1}\zeta^\alpha\widehat{F}^{(1)}_\alpha(\xi)\Big|^2dS(\xi),
\tag{13.27}
$$

where we are taking the standard Fourier transform and $dS(\xi)$ is the standard measure on a sphere. (13.27) is equivalent to the linear independence of the finite family of functions $\{\widehat{F}_\alpha^{(1)}\}_\alpha$ on the sphere of radius $\sqrt{(N+1)\lambda_0 - \omega_0}$. This independence is in general expected to be true. This point is discussed in [GW] for a special situation involving small ground states and $N = 1$.

References

[]BC D.Bambusi, S.Cuccagna, *On dispersion of small energy solutions of the nonlinear Klein Gordon equation with a potential*, Amer. Math. Jour., 133 (2011), 1421–1468 .

[]Be M.Beceanu, *A Critical Centre-Stable Manifold for Schrödinger's Equation in $\mathbb{R}^3$* , arXiv:0909.1180.

[]BuC N.Boussaid, S.Cuccagna, *On stability of standing waves of nonlinear Dirac equations*, arXiv:1103.4452.

[]BP1 V.Buslaev, G.Perelman, *Scattering for the nonlinear Schrödinger equation: states close to a soliton*, St. Petersburg Math.J., 4 (1993), 1111–1142.

[]BP2 V.Buslaev, G.Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations, editor N.N. Uraltseva, Transl. Ser. 2, 164, Amer. Math. Soc., 75–98, Amer. Math. Soc., Providence (1995).

[]BS V.Buslaev, C.Sulem, *On the asymptotic stability of solitary waves of Nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré. An. Nonlin., 20 (2003), 419–475.

[]CL T.Cazenave, P.L.Lions, *Orbital stability of standing waves for nonlinear Schrödinger equations* , Comm. Math. Phys. 85 (1982), 549–561.

[]Cu1 S.Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states*, Comm. Math. Physics, 305 (2011), 279-331.

[]Cu2 S.Cuccagna, *On asymptotic stability of ground states of NLS*, Rev. Math. Phys. 15 (2003), 877–903.

[]Cu3 S.Cuccagna, *On instability of excited states of the nonlinear Schrödinger equation*, Physica D, 238 (2009), 38–54.

[]Cu4 S. Cuccagna, *On scattering of small energy solutions of non autonomous hamiltonian nonlinear Schrödinger equations*, J. Differential Equations, 250 (2011), 2347-2371.

[]Cu5 S.Cuccagna, *On asymptotic stability in energy space of ground states of NLS in 1D*, J. Diff. Eq., 245 (2008), 653-691

[]Cu6 S.Cuccagna, *Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure App. Math. 54 (2001), pp. 1110–1145, erratum Comm. Pure Appl. Math. 58 (2005), p. 147.

[]CM S.Cuccagna, T.Mizumachi, *On asymptotic stability in energy space of ground states for Nonlinear Schrödinger equations*, Comm. Math. Phys., 284 (2008), 51–87.

[]CPV S.Cuccagna, D.Pelinovsky, V.Vougalter, *Spectra of positive and negative energies in the linearization of the NLS problem*, Comm. Pure Appl. Math. 58 (2005), 1–29.

[]GS Zhou Gang, I.M.Sigal, *Relaxation of Solitons in Nonlinear Schrödinger Equations with Potential* , Advances in Math., 216 (2007), 443-490.

[]GW Zhou Gang, M.I.Weinstein, *Dynamics of Nonlinear Schrödinger/Gross-Pitaeskii Equations; Mass transfer in Systems with Solitons and Degenerate Neutral Modes*, Anal. PDE 1 (2008), 267–322.

[]GSS1 M.Grillakis, J.Shatah, W.Strauss, *Stability of solitary waves in the presence of symmetries, I* , Jour. Funct. An. 74 (1987), 160–197.

[]GSS2 M.Grillakis, J.Shatah, W.Strauss, *Stability of solitary waves in the presence of symmetries, II*, Jour. Funct. An. 94 (1990), 308–348.

[]KW E.Kirr, M.I.Weinstein *Diffusion of power in randomly perturbed Hamiltonian partial differential equations*, Comm. Math. Phys. 255 (2005), no. 2, 293–328.

[]MS H.McKean, J.Shatah *The nonlinear Schrödinger equation and the nonlinear heat equation reduction to linear form* , Comm. Pure App. Math. 44 (1991), 10671080.

[]N E. Nelson *Topics in dynamics I: Flows* , Princeton University Press, 1969.

[]O P.J.Olver *Applications of Lie groups to differential equations*. Second edition. Graduate Texts in Mathematics, 107. Springer-Verlag, New York, 1993.

[]P G.Perelman, *Asymptotic stability of solitons for nonlinear Schrödinger equations*, Comm. in PDE 29 (2004), 1051–1095.

[]RSS I.Rodnianski, W.Schlag, A.Soffer, *Asymptotic stability of N-soliton states of NLS* , (2003), arXiv:math/0309114v1.

[]S J.Shatah, *Stable standing waves of nonlinear Klein-Gordon equations* , Comm. Math. Phys., 91 (1983), no. 3, 313327.

[]SW1 A.Soffer, M.I.Weinstein, *Multichannel nonlinear scattering for nonintegrable equations* , Comm. Math. Phys., 133 (1990), 116–146.

[]SW2 A.Soffer, M.I.Weinstein, *Multichannel nonlinear scattering II. The case of anisotropic potentials and data* , J. Diff. Eq., 98 (1992), 376–390.

[]SW3 A.Soffer, M.I.Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations* , Invent. Math., 136 (1999), 9–74.

[]Si I.M.Sigal, *Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasi- periodic solutions* , Comm. Math. Phys. 153 (1993), 297–320.

[]Ta M.Taylor, *Partial differential equations.*, Appl. Math. Sci. vol. 117, Springer-Verlag, New York, 1996.

[]Ts T.P.Tsai, *Asymptotic dynamics of nonlinear Schrödinger equations with many bound states*, J. Diff. Eq. 192 (2003), 225–282.

[]TY1 T.P.Tsai, H.T.Yau, *Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and radiation dominated solutions*, Comm. Pure Appl. Math. 55 (2002), 153–216.

[]TY2 T.P.Tsai, H.T.Yau, *Relaxation of excited states in nonlinear Schrödinger equations*, Int. Math. Res. Not. 31 (2002), 1629–1673.

[]TY3 T.P.Tsai, H.T.Yau, *Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data*, Adv. Theor. Math. Phys. 6 (2002), 107–139.

[]W1 M.I.Weinstein, *Lyapunov stability of ground states of nonlinear dispersive equations*, Comm. Pure Appl. Math. 39 (1986), 51–68.

[]W2 M.I.Weinstein, *Modulation stability of ground states of nonlinear Schrödinger equations*, Siam J. Math. Anal. 16 (1985), 472–491.

Department of Mathematics and Computer Sciences, University of Trieste, via Valerio 12/1 Trieste, 34127 Italy

E-mail Address: `scuccagna@units.it`